M.Sc. II Sem. (Mathematics)

Paper 1st - Advanced Abstract Algebra-II

Unit - III

Reference Book : P.B.Bhattacharya, S.K. Jain and S.R Nagpaul, *Basic Abstract Algebra* (2nd Edition), Cambridge University Press, Indian Edition, 1997.

Topic : Noetherian and Artinian Modules

Definition : An R-module M is said to satisfy the **Ascending Chain Condition (ACC)** if given any chain $M_1 \subseteq M_2 \subseteq \dots \subseteq M_{n+1} \subseteq \dots$ of submodules of M, there exists $m \in N$ such that $M_n = M_m$ for all $n \geq m$.

Definition : An R-module M is said to satisfy the **Descending Chain Condition (DCC)** if given any chain $M_1 \supseteq M_2 \supseteq \dots \supseteq M_{n+1} \supseteq \dots$ of submodules of M, there exists $m \in N$ such that $M_n = M_m$ for all $n \geq m$.

Definition : An R-module M is called **Noetherian** if for every ascending sequence of R-submodules of M, $M_1 \subset M_2 \subset M_3 \subset \dots$ there exists a postive integer k such that $M_k = M_{k+1} = M_{k+2} = \dots.$

In other words, if M is noetherian , then we say that the ascending chain condition for submodules holds in M.

Or

An R-module M is noetherian if M has ACC for submodules of M.

Definition : An R-module M is called **Artinian** if for every descending sequence of R-submodules of M, $M_1 \supset M_2 \supset M_3 \supset \dots$ there exists a postive integer k such that $M_k = M_{k+1} = M_{k+2} = \dots.$

In other words, if M is artinian, then we say that the descending chain condition for submodules holds in M.

Or

An R-module M is artinian if M has DCC for submodules of M.

Theorem. For an R-module M, the following are equivalent :

(i) M is noetherian.

(ii) Every submodule of M is finitely generated.

(iii) Every non-empty set S of submodules of M has a maximal element (i.e. a submodule M_0 in S such that for any submodule N_0 in S with $N_0 \supset M_0$, we have $N_0 = M_0$).

Proof. (i) ⇒ (ii) : Suppose M is noetherian, i.e., M has ascending chain condition for submodules. Then we have to show that every submodule of M is finitely generated.

Suppose N be a submodule of M.

Suppose on the contrary that N is not finitely generated.

For any positive integer k, suppose $a_1, a_2, \ldots, a_k \in N$, then

$$N \neq (a_1, a_2, \ldots, a_k).$$

Choose $a_{k+1} \in N$ such that

$$a_{k+1} \notin (a_1, a_2, \ldots, a_k)$$

We then obtain an infinite properly ascending chain

$$a_1 \nsubseteq (a_1, a_2) \nsubseteq (a_1, a_2, a_3) \nsubseteq \ldots \nsubseteq (a_1, \ldots, a_{k+1}) \nsubseteq \ldots$$

of submodules of M, which is a contradiction.

Hence, N is finitely generated.

Since N is arbitrary, hence, every submodule of M is finitely generated.

(ii) ⇒ (iii) : Suppose every submodule of M is finitely generated. Then we have to show that every non-empty set S of submodules of M has a maximal element.

Let N_0 be an element of S.

If N_0 is not maximal, it is properly contained in a submodule $N_1 \in S$.

If N_1 is not maximal, then N_1 is properly contained in a submodule $N_2 \in S$.

In case, S has no maximal element, we obtain an infinite properly ascending chain of submodules $N_0 \subset N_1 \subset N_2 \subset \ldots$ of M.

Let $N = \bigcup_i N_i$.

Then we have to show that N is submodule of M.

Consider $x, y \in \bigcup_i N_i$ and $r \in R$.

Then, $x \in N_\mu$ and $y \in N_\nu$ for some μ and ν.

Since either $N_\mu \subset N_\nu$ or $N_\nu \subset N_\mu$ and so both x and y lie in one submodule N_μ or N_ν.

And hence $x - y$ and $r.x$ lie in one submodule.

This implies that $x - y \in N$ and $r.x \in N$.

Hence, N is submodule of M.

⇒ N is finitely generated. (By hypothesis)

So there exists elements $a_1, a_2, \ldots, a_n \in N$ such that

$N = (a_1, a_2, \ldots, a_n)$.

Now, $a_1, a_2, \ldots, a_n$ belongs to a finite number of submodules N_i, $i = 1, 2, \ldots$.

Hence, there exists N_k such that all a_i, $1 \leq i \leq n$ lie in a_k.

Since $N_k \subset N$ and N is the smallest submodule containing all a_i, $1 \leq i \leq n$, it follows that $N_k = N$.

But then $N_k = N_{k+1} = N_{k+2} = \ldots = N$, which is a contradiction, since S has no maximal element.

Hence, S has a maximal element.

(iii) $\Rightarrow$ (i) : Suppose every non-empty set of submodules of M has a maximal element. Then we have to show that M is noetherian.

Suppose we have an ascending sequence of submodules of M,

$$M_1 \subset M_2 \subset M_3 \subset \ldots$$

By hypothesis, the sequence $M_1, M_2, \ldots$ has a maximal element, say, M_k.

Then $M_k = M_{k+1} = M_{k+2} = \ldots$ for $k \geq 1$.

Hence, M has ACC for submodules.

Thus, M is noetherian.

Hence proved.

Theorem : Every submodule of a noetherian module is also a noetherian module.

Proof. Suppose M be a noetherian module and suppose L be a submodule of M.

Then we have to show that L is noetherian.

Suppose U be a submodule of L, then U is a also a submodule of M.

It follows that U is finitely generated. (∵ If M is noetherian, then every submodule of M is finitely generated)

Since U is arbitrary and so every submodule of L is finitely generated.

Hence, L is a noetherian module. (∵ If every submodule of M is finitely generated, then M is noetherian)

Since L is arbitrary and so every submodule of a noetherian module is noetherian.

Hence proved.