UNIT I: INTRODUCTION TO DIGITAL SIGNAL PROCESSING

1.1 INTRODUCTION

Signals constitute an important part of our daily life. Anything that carries some information is called a signal. A signal is defined as a single-valued function of one or more independent variables which contain some information. A signal is also defined as a physical quantity that varies with time, space or any other independent variable. A signal may be represented in time domain or frequency domain. Human speech is a familiar example of a signal. Electric current and voltage are also examples of signals. A signal can be a function of one or more independent variables. A signal may be a function of time, temperature, position, pressure, distance etc. If a signal depends on only one independent variable, it is called a one- dimensional signal, and if a signal depends on two independent variables, it is called a two- dimensional signal.

A system is defined as an entity that acts on an input signal and transforms it into an output signal. A system is also defined as a set of elements or fundamental blocks which are connected together and produces an output in response to an input signal. It is a cause-and- effect relation between two or more signals. The actual physical structure of the system determines the exact relation between the input $x(n)$ and the output $y(n)$, and specifies the output for every input. Systems may be single-input and single-output systems or multi-input and multi-output systems.

Signal processing is a method of extracting information from the signal which in turn depends on the type of signal and the nature of information it carries. Thus signal processing is concerned with representing signals in the mathematical terms and extracting information by carrying out algorithmic operations on the signal. Digital signal processing has many advantages over analog signal processing. Some of these are as follows:

Digital circuits do not depend on precise values of digital signals for their operation. Digital circuits are less sensitive to changes in component values. They are also less sensitive to variations in temperature, ageing and other external parameters.

In a digital processor, the signals and system coefficients are represented as binary words. This enables one to choose any accuracy by increasing or decreasing the number of bits in the binary word.

Digital processing of a signal facilitates the sharing of a single processor among a number of signals by time sharing. This reduces the processing cost per signal.

Digital implementation of a system allows easy adjustment of the processor characteristics during processing.

Linear phase characteristics can be achieved only with digital filters. Also multirate processing is possible only in the digital domain. Digital circuits can be connected in cascade without any loading problems, whereas this cannot be easily done with analog circuits.

Storage of digital data is very easy. Signals can be stored on various storage media such as magnetic tapes, disks and optical disks without any loss. On the other hand, stored analog signals deteriorate rapidly as time progresses and cannot be recovered in their original form.

Digital processing is more suited for processing very low frequency signals such as seismic signals.

Though the advantages are many, there are some drawbacks associated with processing a signal in digital domain. Digital processing needs ‘pre’ and ‘post’ processing devices like analog-to-digital and digital-to-analog converters and associated reconstruction filters. This increases the complexity of the digital system. Also, digital techniques suffer from frequency limitations. Digital systems are constructed using active devices which consume power whereas analog processing algorithms can be implemented using passive devices which do not consume power. Moreover, active devices are less reliable than passive components. But the advantages of digital processing techniques outweigh the disadvantages in many applications. Also the cost of DSP hardware is decreasing continuously. Consequently, the applications of digital signal processing are increasing rapidly.

The digital signal processor may be a large programmable digital computer or a small microprocessor programmed to perform the desired operations on the input signal. It may also be a hardwired digital processor configured to perform a specified set of operations on the input signal.

DSP has many applications. Some of them are: Speech processing, Communication, Biomedical, Consumer electronics, Seismology and Image processing.

The block diagram of a DSP system is shown in Figure 1.1.

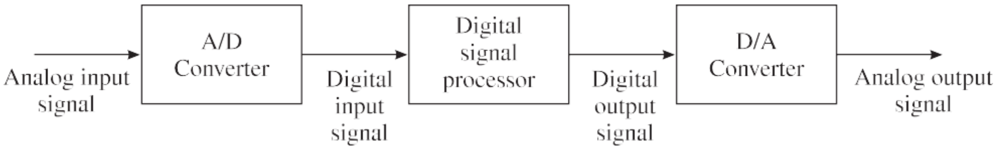


Figure 1.1 Block diagram of a digital signal processing system.

In this book we discuss only about discrete one-dimensional signals and consider only single-input and single-output discrete-time systems. In this chapter, we discuss about various basic discrete-time signals available, various operations on discrete-time signals and classification of discrete-time signals and discrete-time systems.

1.2 REPRESENTATION OF DISCRETE-TIME SIGNALS

Discrete-time signals are signals which are defined only at discrete instants of time. For those signals, the amplitude between the two time instants is just not defined. For discrete- time signal the independent variable is time *n*, and it is represented by x (n).

There are following four ways of representing discrete-time signals:

1. Graphical representation
2. Functional representation
3. Tabular representation
4. Sequence representation

1.2.1 Graphical Representation

Consider a single x (n) with values

X (-2) = -3, x(-1) = 2, x(0) = 0, x(1) = 3, x(2) = 1 and x(3) = 2

This discrete-time single can be represented graphically as shown in Figure 1.2

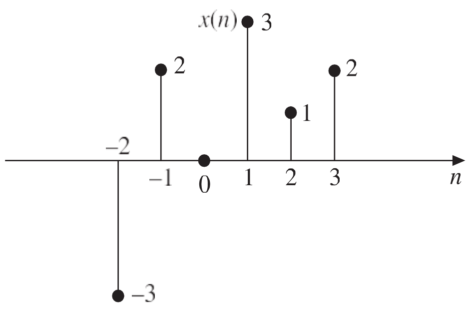


Figure 1.2 Graphical representation of discrete-time signal

1.2.2 Functional Representation

In this, the amplitude of the signal is written against the values of *n*. The signal given in section 1.2.1 can be represented using the functional representation as follows:

$$x(n) = \begin{cases} -3 & \text{for } n = -2 \\ 2 & \text{for } n = -1 \\ 0 & \text{for } n = 0 \\ 3 & \text{for } n = 1 \\ 1 & \text{for } n = 2 \\ 2 & \text{for } n = 3 \end{cases}$$

Another example is:

$$X(n) = 2^n u(n)$$

Or $$x(n) = \begin{cases} 2^n & for\ n \geq 0 \\ 0 & for\ n < 0 \end{cases}$$

1.2.3 Tabular Representation

In this, the sampling instant *n* and the magnitude of the signal at the sampling instant are represented in the tabular form. The signal given in section 1.2.1 can be represented in tabular form as follows:

n	2	1	0	1	2	3
x (*n*)	3	2	0	3	1	2

1.2.4 Sequence Representation

A finite duration sequence given in section 1.2.1 can be represented as follows:

$$X(n) = \left\{ \underset{\uparrow}{-3,2,0,3,1,2} \right\}$$

Another example is:

$$X(n) = \left\{ \underset{\uparrow}{\ldots 2,3,0,1,-2 \ldots} \right\}$$

The arrow mark ↑ denotes the $n = 0$ term. When no arrow is indicated, the first term corresponds to $n = 0$.
So a finite duration sequence, that satisfies the condition $x(n) = 0$ for $n < 0$ can be represented as:

$$x(n) = \{3, 5, 2, 1, 4, 7\}$$

SuN and product of discrete-tiNe sequences

The sum of two discrete-time sequences is obtained by adding the corresponding elements of sequences

$$\{C_n\} = \{a_n\} + \{b_n\} \rightarrow C_n = a_n + b_n$$

The product of two discrete-time sequences is obtained by multiplying the corresponding elements of the sequences.

$$\{C_n\} = \{a_n\}\{b_n\} \rightarrow C_n = a_n b_n$$

The multiplication of a sequence by a constant *k* is obtained by multiplying each element of the sequence by that constant.

$$\{C_n\} = k\{a_n\} \rightarrow C_n = k a_n$$

1.3 ELEMENTARY DISCRETE-TIME SIGNALS

There are several elementary signals which play vital role in the study of signals and systems. These elementary signals serve as basic building blocks for the construction of more complex signals. Infact, these elementary signals may be used to model a large number of physical signals, which occur in nature. These elementary signals are also called standard signals.

The standard discrete-time signals are as follows:

1. Unit step sequence
2. Unit ramp sequence
3. Unit parabolic sequence
4. Unit impulse sequence
5. Sinusoidal sequence
6. Real exponential sequence
7. Complex exponential sequence

1.3.1 Unit Step Sequence

The step sequence is an important signal used for analysis of many discrete-time systems. It exists only for positive time and is zero for negative time. It is equivalent to applying a signal whose amplitude suddenly changes and remains constant at the sampling instants forever after application. In between the discrete instants it is zero. If a step function has unity magnitude, then it is called unit step function.

The usefulness of the unit-step function lies in the fact that if we want a sequence to start at $n = 0$, so that it may have a value of zero for $n < 0$, we only need to multiply the given sequence with unit step function $u(n)$.

The discrete-time unit step sequence $u(n)$ is defined as:

$$\mathrm{U}(\mathrm{n}) = \begin{cases} 1 \; for\; n \geq 0 \\ 0 \; for\; n < 0 \end{cases}$$

The shifted version of the discrete-time unit step sequence $u(n - k)$ is defined as:

$$\mathrm{U}(\mathrm{n} - \mathrm{k}) = \begin{cases} 1 \; for\; n \geq k \\ 0 \; for\; n < k \end{cases}$$

It is zero if the argument $(n - k) < 0$ and equal to 1 if the argument $(n - k)$ S 0.

The graphical representation of $u(n)$ and $u(n - k)$ is shown in Figure 1.3[(a) and (b)].

Figure 1.3 Discrete-time (a) Unit step function (b) Shifted unit step function

1.3.2 Unit Ramp Sequence

The discrete-time unit ramp sequence $r(n)$ is that sequence which starts at $n = 0$ and increases linearly with time and is defined as:

$$\mathrm{r}(\mathrm{n}) = \begin{cases} n \; for\; n \geq 0 \\ 0 \; for\; n < 0 \end{cases}$$

or $$r(n) = nu(n)$$

It starts at $n = 0$ and increases linearly with n.

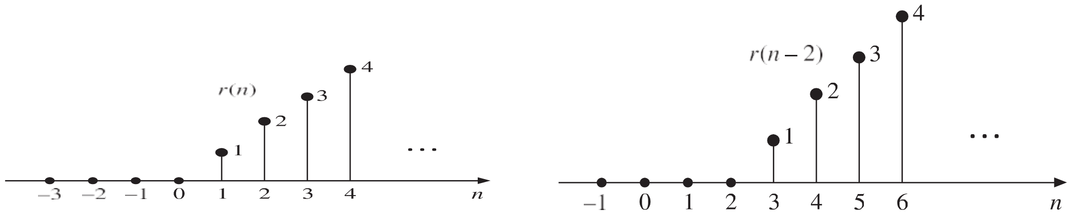
The shifted version of the discrete-time unit ramp sequence $r(n - k)$ is defined as:

$$\mathrm{R}(\mathrm{n} - \mathrm{k}) = \begin{cases} n - k \; for\; n \geq k \\ 0 \quad\quad for\; n < k \end{cases}$$

Or $$r(n - k) = (n - k)\, u(n - k)$$

The graphical representation of $r(n)$ and $r(n-2)$ is shown in Figure 1.4[(a) and (b)].

Figure 1.4 Discrete-time (a) Unit ramp sequence (b) Shifted ramp sequence.



1.3.3 Unit Parabolic Sequence

The discrete-time unit parabolic sequence $p\ (n)$ is defined as:

$$P\ (n) = \begin{cases} \frac{n^2}{2} & for\ n\ \geq 0 \\ 0 & for\ n\ < 0 \end{cases}$$

Or $$P(n) = \frac{N^2}{2}\ u(n)$$

The shifted version of the discrete-time unit parabolic sequence $p(n-k)$ is defined as:

$$P(n-k) = \begin{cases} \frac{(n-k)^2}{2} & for\ n\ \geq k \\ 0 & for\ n\ <\ k \end{cases}$$

Or $$p(n-k) = \frac{(n-k\ )^2}{2}\ u(n-k)$$

The graphical representation of $p(n)$ and $p(n-3)$ is shown in Figure 1.5[(a) and (b)].

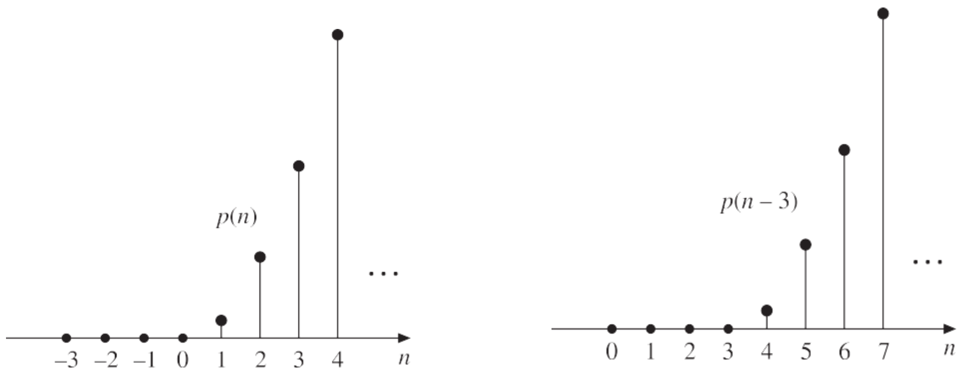


Figure 1.5 Discrete-time (a) ParaboFic sequence (b) Shifted paraboFic sequence.

1.3.4 Unit Impulse Function or Unit Sample Sequence

The discrete-time unit impulse function (n), also called unit sample sequence, is defined as:

$$\delta\ (n) = \begin{cases} 1\ for\ n = 0 \\ 0\ for\ n\ \neq 0 \end{cases}$$

This means that the unit sample sequence is a signal that is zero everywhere, except at $n = 0$, where its value is unity. It is the most widely used elementary signal used for the analysis of signals and systems.

The shifted unit impulse function ($n-k$) is defined as:

$$\delta\ (n-k) = \begin{cases} 1\ for\ n = k \\ 0\ for\ n\ \neq k \end{cases}$$

The graphical representation of (n) and ($n-k$) is shown in Figure 1.6[(a) and (b)].

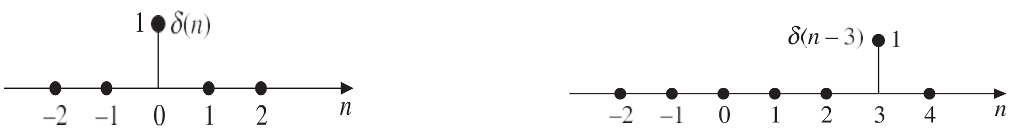


Figure 1.6 Discrete-time (a) Unit sample sequence (b) Delayed unit sample sequence.

Properties of discrete-time unit sample sequence

1. δ(n) = u(n) – u(n – 1)
2. δ(n – k) = $\begin{cases} 1 \; for \; n = k \\ 0 \; for \; n \neq k \end{cases}$
3. X(n) = $\sum_{k=-\infty}^{\infty} x(k)\delta\,(n-k)$
4. $\sum_{n=-\infty}^{\infty} x(n)\delta\,(n-\,n_0)$ = x(n_0)

Relation Between The Unit Sample Sequence And The Unit Step Sequence

The unit sample sequence $\delta(n)$ and the unit step sequence $u(n)$ are related as:

$$U(n) = \sum_{m=0}^{n} \delta\,(m),\ \delta(n) = u(n) - u(n - 1)$$

Sinusoidal Sequence

The discrete-time sinusoidal sequence is given by

$$X(n) = A \sin(\omega n + \emptyset)$$

Where *A* is the amplitude, is angular frequency, is phase angle in radians and *n* is an integer.

The period of the discrete-time sinusoidal sequence is:

$$N = \frac{2\pi}{\omega}\, m$$

Where *N* and *m* are integers.

All continuous-time sinusoidal signals are periodic, but discrete-time sinusoidal sequences may or may not be periodic depending on the value of.

For a discrete-time signal to be periodic, the angular frequency must be a rational multiple of 2. The graphical representation of a discrete-time sinusoidal signal is shown in Figure 1.7.

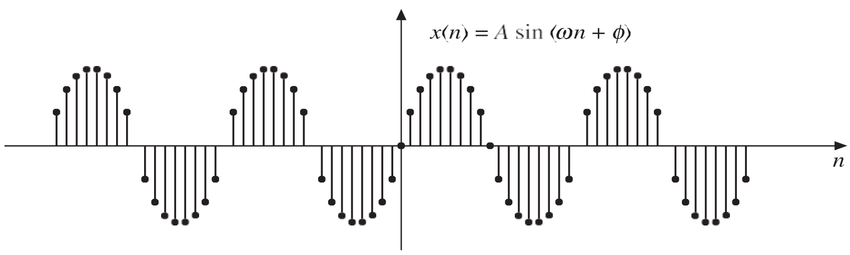


Figure 1.7 Discrete-time sinusoidal signal

1.3.6 Real Exponential Sequence

The discrete-time real exponential sequence a^n is defined as:

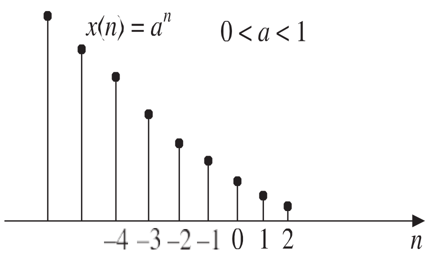
$$X(n) = a^n \text{ for all n}$$

Figure 1.8 illustrates different types of discrete-time exponential signals.

When *a* > 1, the sequence grows exponentially as shown in Figure 1.8(a).

When 0 < *a* < 1, the sequence decays exponentially as shown in Figure 1.8(b).

When *a* < 0, the sequence takes alternating signs as shown in Figure 1.8[(c) and



(d)].

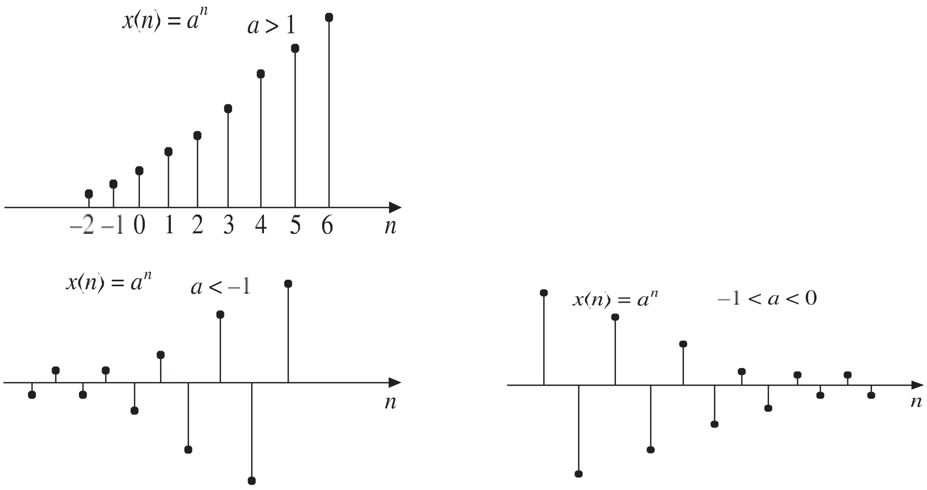


Figure 1.8 Discrete-time exponential signal an for (a) a > 1 (b) 0 < a < 1 (c) a < -1 (d) -1 < a < 0.

1.3.7 Complex Exponential Sequence

The discrete-time complex exponential sequence is defined as:

$$X(n) = a^n e^{j(\omega_0 n + \emptyset)}$$

$$= an \cos(\omega_0 n + \emptyset) + ja^n \sin(\omega_0 n = \emptyset)$$

For $|a| = 1$, the real and imaginary parts of complex exponential sequence are sinusoidal.

For $|a| > 1$, the amplitude of the sinusoidal sequence exponentially grows as shown in Figure 1.9(a).

For $|a| < 1$, the amplitude of the sinusoidal sequence exponentially decays as shown in Figure 1.9(b).

EXAMPLE 1.1 Find the following summations:

(a) $\sum_{n=-\infty}^{\infty} e^{3n}\,\delta(n-3)$ (b) $\sum_{n=-\infty}^{\infty} \delta(n-2)\cos 3n$

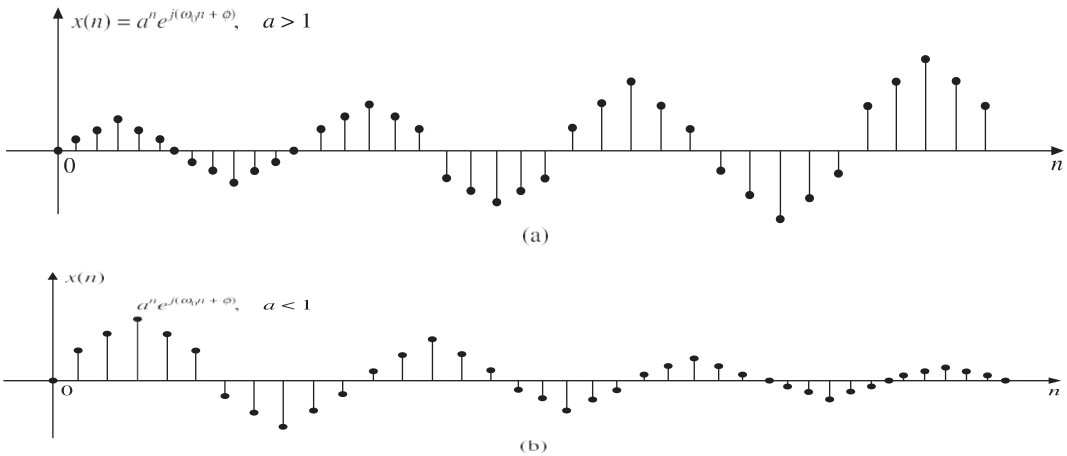


Figure 1.9 complex exponential sequence $x(n) = a^n e^j(\omega_0 n + \emptyset)$ for (a) a > 1 (b) a < 1.

(c) $\sum_{n=-\infty}^{\infty} n^2\,\delta(n+4)$ (d) $\sum_{n=-\infty}^{\infty} \delta\ (n-2)\,e^{n^2}$

(e) $\sum_{n=0}^{\infty} \delta(n+1)\ 4^n$

Solution:

(a) Given $\boldsymbol{\sum_{n=-\infty}^{\infty} \delta^{3n}\,\delta(n-3)}$

We know that $\delta(n-3) = \begin{cases} 1 & for\ n=3 \\ 0 & elsewhere \end{cases}$

$$\sum_{n=-\infty}^{\infty} e^{3n}\,\delta\,(n-3) = [e^{3n}]_{n=3} = e^{9}$$

(a) Given $\sum_{n=-\infty}^{\infty} \delta(n-2)\cos 3n$

We know that $\delta(n-2) = \begin{cases} 1 & for\ n=2 \\ 0 & elsewhere \end{cases}$

$$\sum_{n=-\infty}^{\infty} \delta(n-2) cos 3n = [\cos 3n]_{n=2} = \cos 6$$

(b) Given $\sum_{n=-\infty}^{\infty} n^2\,\delta(n=4)$

We know that $\delta(n=4) = \begin{cases} 1 & for\ n=-4 \\ 0 & elsewhere \end{cases}$

$$\sum_{n=n-\infty}^{\infty} n^2 \delta(n=4) = [n^2]_{n=-4} = 16$$

(c) Given $\sum_{n=-\infty}^{\infty} \delta(n-2) e^{n^2}$

We know that $\delta(n-2) = \begin{cases} 1\ for\ n=2 \\ 0\ elsewhere \end{cases}$

$$\sum_{n=-\infty}^{\infty} \delta(n-2) e^{n^2} = [e^{n^2}]_{n=2} = e^{2^2} = e^4$$

(d) Given $\sum_{n=0}^{\infty} \delta(n=1) 4^n$

We know that $\delta(n=1) = \begin{cases} 1\ for\ n=-1 \\ 0\ for\ n \neq\ -1 \end{cases}$

$$\sum_{n=0}^{\infty} \delta(n+1) 4^n = 0$$

1.4 BASIC OPERATIONS ON SEQUENCES

When we process a sequence, this sequence may undergo several manipulations involving the independent variable or the amplitude of the signal.

The basic operations on sequences are as follows:

1. Time shifting
2. Time reversal
3. Time scaling
4. Amplitude scaling
5. Signal addition
6. Signal multiplication

The first three operations correspond to transformation in independent variable *n* of a signal. The last three operations correspond to transformation on amplitude of a signal.

1.4.1 Time Shifting

The time shifting of a signal may result in time delay or time advance. The time shifting operation of a discrete-time signal $x(n)$ can be represented by

$$y(n) = x(n-k)$$

This shows that the signal $y(n)$ can be obtained by time shifting the signal $x(n)$ by k units. If k is positive, it is delay and the shift is to the right, and if k is negative, it is advance and the shift is to the left.

An arbitrary signal $x(n)$ is shown in Figure 1.10(a). $x(n-3)$ which is obtained by shifting $x(n)$ to the right by 3 units (i.e. delay $x(n)$ by 3 units) is shown in Figure 1.10(b). $x(n+2)$ which is obtained by shifting $x(n)$ to the left by 2 units (i.e. advancing $x(n)$ by 2 units) is shown in

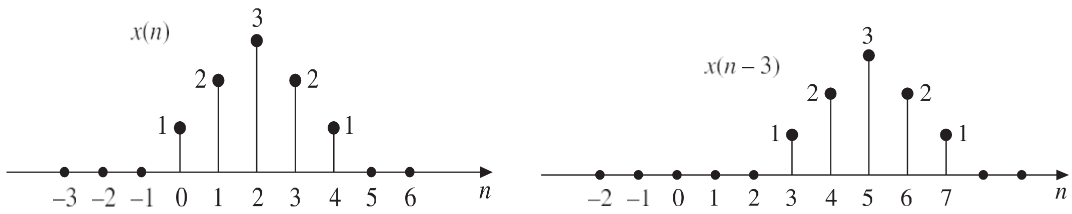


Figure 1.10(c).

Figure 1.10 (a) Sequence x(n) (b) x(n – 3) (c) x(n + 2).

1.4.2 Time Reversal

The time reversal also called time folding of a discrete-time signal $x(n)$ can be obtained by foldingthe sequence about $n = 0$. The time reversed signal is the reflection of the original signal. It is obtained by replacing the independent variable n by $-n$. Figure 1.11(a) shows an arbitrary discrete-time signal $x(n)$, and its time reversed version $x(-n)$ is shown in Figure 1.11(b).

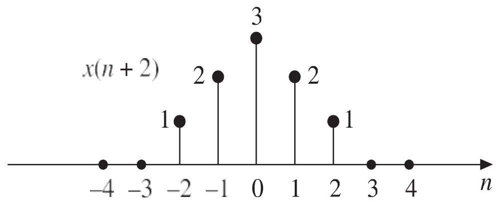


Figure 1.11[(c) and (d)] shows the delayed and advanced versions of reversed signal $x(-n)$.

The signal $x(-n+3)$ is obtained by delaying (shifting to the right) the time reversed signal $x(-n)$ by 3 units of time. The signal $x(-n-3)$ is obtained by advancing (shifting to the left) the time reversed signal $x(-n)$ by 3 units of time.

Figure 1.12 shows other examples for time reversal of signals

EXAMPLE 1.2 Sketch the following signals:

(a) U(n+2) u(-n+3)　　　　(b) x(n) = u(n+4) – u(n-2)

Solutions:

(a) Given　　x(n)=u(n+2) u(-n+3)

The signal $u(n+2)\,u(-n+3)$ can be obtained by first drawing the signal $u(n+2)$ as shown in Figure 1.13(a), then drawing $u(-n+3)$ as shown in Figure 1.13(b),

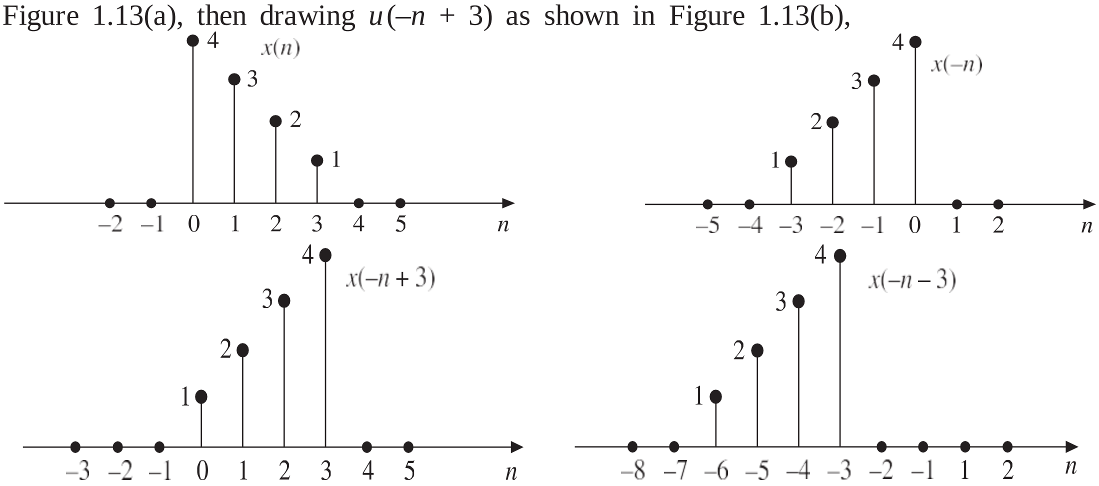


Figure 1.11 (a) Original signal x(n) (b) Time reversed signal x(-n) (c) Time reversed and delayed

signal x(-n+3) (d) Time reversed and advanced signal x(-n-3).

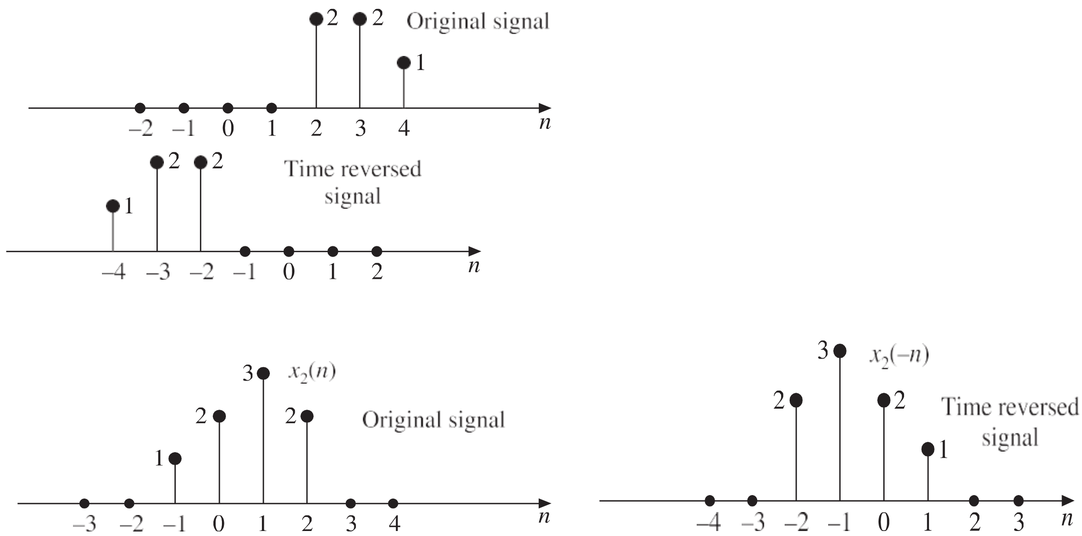


Figure 1.12 Time reversal operations.

and then multiplying these sequences element by element to obtain $u(n + 2)\ u(-n + 3)$ as shown in Figure 1.13(c).

$$x(n) = 0 \quad \text{for} \quad n < -2 \quad \text{and} \quad n > 3;\ x(n) = 1 \quad \text{for} \quad -2 < n < 3$$

(a) Given $$x(n) = u(n + 4) - u(n - 2)$$

The signal $u(n + 4) - u(n - 2)$ can be obtained by first plotting $u(n + 4)$ as shown in Figure 1.14(a), then plotting $u(n - 2)$ as shown in Figure 1.14(b), and then subtracting each element of $u(n - 2)$ from the corresponding element of $u(n + 4)$ to obtain the result shown in Figure 1.14(c).

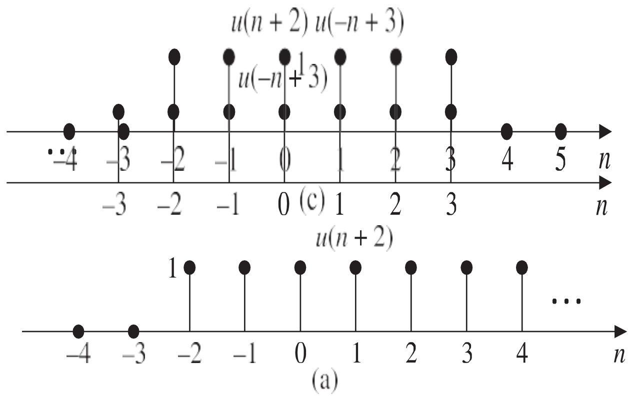
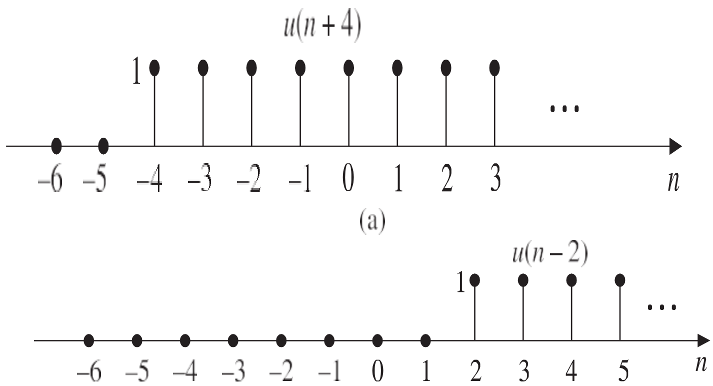


Figure 1.13 Plots of (a) u(n + 2) (b) u(–n + 3) (c) u(n + 2) u(–n + 3).



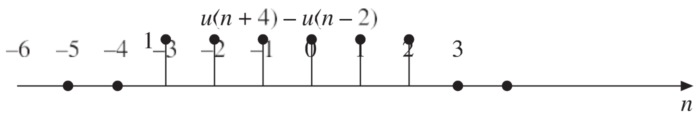


Figure 1.14 Plots of (a) u(n + 4) (b) u(n – 2) (c) u(n + 4) – u(n – 2).

1.4.3 Amplitude Scaling

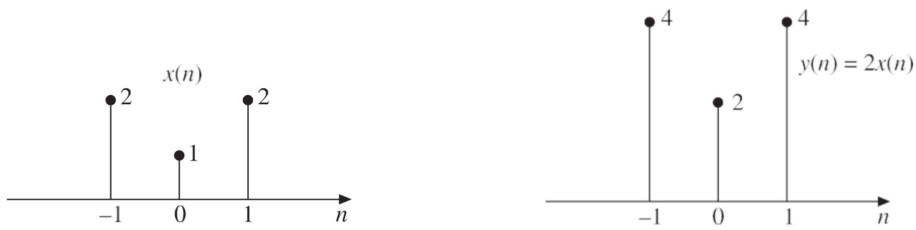
The amplitude scaling of a discrete-time signal can be represented by

$$y(n) = ax(n)$$

where a is a constant.

The amplitude of $y(n)$ at any instant is equal to a times the amplitude of $x(n)$ at that instant. If $a > 1$, it is amplification and if $a < 1$, it is attenuation. Hence the amplitude is rescaled. Hence the name amplitude scaling.

Figure 1.15(a) shows a signal $x(n)$ and Figure 1.15(b) shows a scaled signal $y(n) = 2x(n)$.



1.4.1 Time Scaling

Time scaling may be time expansion or time compression. The time scaling of a discrete- time signal $x(n)$ can be accomplished by replacing n by an in it. Mathematically, it can be expressed as:

$$y(n) = x(an)$$

When $a > 1$, it is time compression and when $a < 1$, it is time expansion.

Let $x(n)$ be a sequence as shown in Figure 1.16(a). If $a = 2$, $y(n) = x(2n)$. Then

$$y(0) = x(0) = 1$$
$$y(-1) = x(-2) = 3$$
$$y(-2) = x(-4) = 0$$
$$y(1) = x(2) = 3$$
$$y(2) = x(4) = 0$$

and so on.

So to plot $x(2n)$ we have to skip odd numbered samples in $x(n)$.

We can plot the time scaled signal $y(n) = x(2n)$ as shown in Figure 1.16(b). Here the signal is compressed by 2.

If $a = (1/2)$, $y(n) = x(n/2)$, then

$$y(0) = x(0) = 1$$
$$y(2) = x(1) = 2$$
$$y(4) = x(2) = 3$$
$$y(6) = x(3) = 4$$
$$y(8) = x(4) = 0$$
$$y(-2) = x(-1) = 2$$
$$y(-4) = x(-2) = 3$$
$$y(-6) = x(-3) = 4$$
$$y(-8) = x(-4) = 0$$

We can plot $y(n) = x(n/2)$ as shown in Figure 1.16(c). Here the signal is expanded by 2. All odd components in $x(n/2)$ are zero because $x(n)$ does not have any value in between the sampling instants.

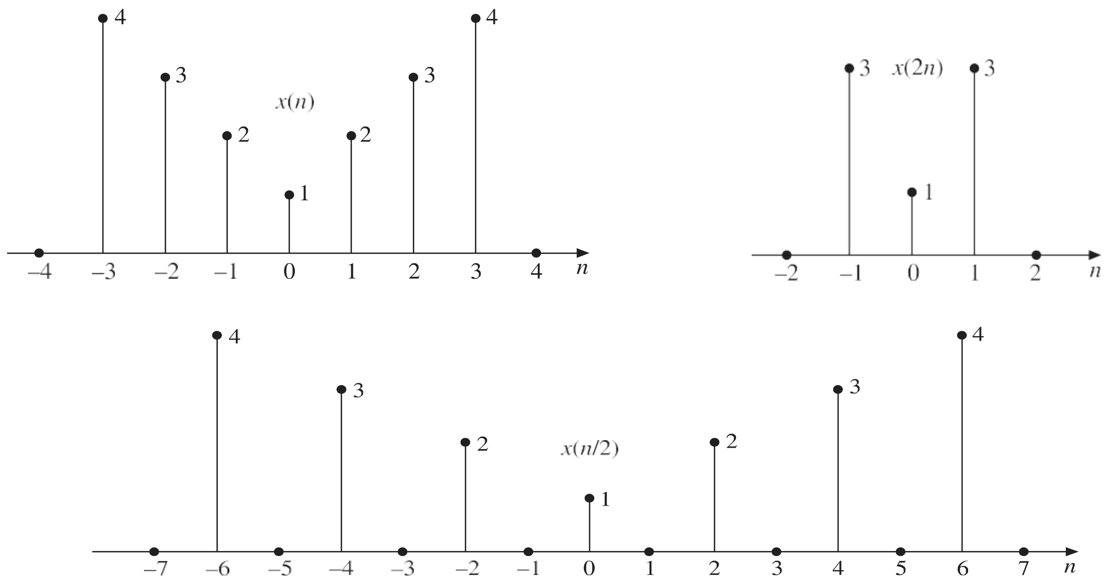


Figure 1.16 Discrete-time scaling (a) Plot of x(n) (b) Plot of x(2n) (c) Plot of x(n/2)

Time scaling is very useful when data is to be fed at some rate and is to be taken out at a different rate.

1.45 Signal Addition

In discrete-time domain, the sum of two signals $x_1(n)$ and $x_2(n)$ can be obtained by adding the corresponding sample values and the subtraction of $x_2(n)$ from $x_1(n)$ can be obtained by subtracting each sample of $x_2(n)$ from the corresponding sample of $x_1(n)$ as illustrated below.

If $x_1(n) = \{1, 2, 3, 1, 5\}$ and $x_2(n) = \{2, 3, 4, 1, -2\}$

Then $x_1(n) + x_2(n) = \{1 + 2, 2 + 3, 3 + 4, 1 + 1, 5 - 2\} = \{3, 5, 7, 2, 3\}$

and $x_1(n) - x_2(n) = \{1 - 2, 2 - 3, 3 - 4, 1 - 1, 5 + 2\} = \{-1, -1, -1, 0, 7\}$

1.4.6 Signal multiplication

The multiplication of two discrete-time sequences can be performed by multiplying their values at the sampling instants as shown below.

If $x_1(n) = \{1, -3, 2, 4, 1.5\}$ and $x_2(n) = \{2, -1, 3, 1.5, 2\}$

Then $x_1(n)\, x_2(n) = \{1 \times 2, -3 \times -1, 2 \times 3, 4 \times 1.5, 1.5 \times 2\}$

$= \{2, 3, 6, 6, 3\}$

EXAMPLE 1.3 Express the signals shown in Figure 1.17 as the sum of singular functions.

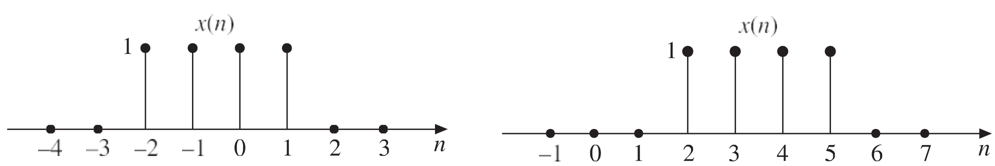


Figure 1.17 Waveforms for Example 1.3

Solution:

(a) The given signal shown in Figure 1.17(a) is:

$$x(n) = \delta(n + 2) + \delta(n + 1) + \delta(n) + \delta(n - 1)$$

$$x(n) = \begin{cases} 0 & for\ n \leq -3 \\ 1 & for -2 \leq n \leq 1 \\ 0 & for\ n \geq 2 \end{cases}$$

$\therefore$ $x(n) = u(n+2) - u(n-2)$

(b) The signal shown in Figure 1.17(b) is:

$$x(n) = \delta(n-2) + \delta\ (n-3) + \delta\ (n-4) + \delta\ (n-5)$$

$$x(n) = \begin{cases} 0 & for\ n\ \leq 1 \\ 1\ for\ 2 \leq n \leq 5 \\ 0 & for\ n \geq 6 \end{cases}$$

$$\therefore \qquad x(n) = u(n-2) - u(n-6)$$

1.4 CLASSIFICATION OF DISCRETE-TIME SIGNALS

The signals can be classified based on their nature and characteristics in the time domain. They are broadly classified as: (i) continuous-time signals and (ii) discrete-time signals.

The signals that are defined for every instant of time are known as continuous-time signals. The continuous-time signals are also called analog signals. They are denoted by *x* (*t*). They are continuous in amplitude as well as in time. Most of the signals available are continuous-time signals.

The signals that are defined only at discrete instants of time are known as discrete-time signals. The discrete-time signals are continuous in amplitude, but discrete in time. For discrete- time signals, the amplitude between two time instants is just not defined. For discrete-time signals, the independent variable is time *n*. Since they are defined only at discrete instants of time, they are denoted by a sequence *x* (*nT*) or simply by *x*(*n*) where *n* is an integer.

Figure 1.18 shows the graphical representation of discrete-time signals. The discrete- time signals may be inherently discrete or may be discrete versions of the continuous-time signals.

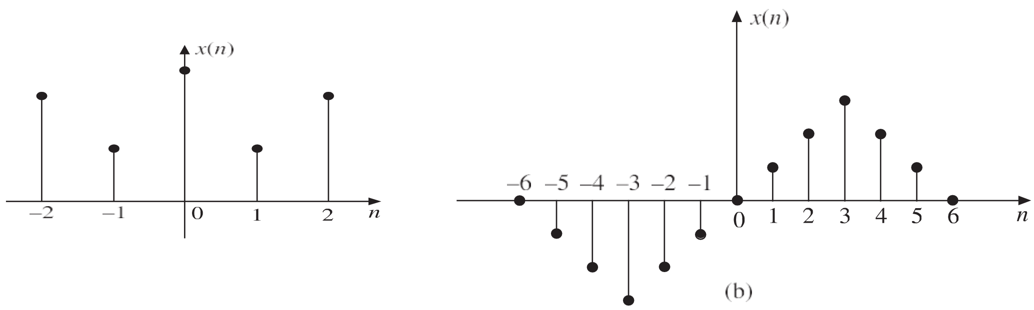


Figure 1.18 Discrete-time signals

Both continuous-time and discrete-time signals are further classified as follows:

1. Deterministic and random signals
2. Periodic and non-periodic signals
3. Energy and power signals
4. Causal and non-causal signals
5. Even and odd signals

1.5.1 Deterministic and Random Signals

A signal exhibiting no uncertainty of its magnitude and phase at any given instant of time is called deterministic signal. A deterministic signal can be completely represented by mathematical equation at any time and its nature and amplitude at any time can be predicted.

Examples: Sinusoidal sequence $x(n) = \cos n$, Exponential sequence $x(n) = e^{j\ n}$, ramp sequence $x(n) = n$.

A signal characterized by uncertainty about its occurrence is called a non-deterministic or random signal. A random signal cannot be represented by any mathematical equation. The behavior of such a signal is probabilistic in nature and can be analyzed only stochastically. The pattern of such a signal is quite irregular. Its amplitude and phase at any time instant cannot be predicted in advance. A typical example of a non-deterministic signal is thermal noise.

1.5.2 Periodic and Non-periodic Sequences

A signal which has a definite pattern and repeats itself at regular intervals of time is called a periodic signal, and a signal which does not repeat at regular intervals of time is called a non-periodic or aperiodic signal.

A discrete-time signal $x(n)$ is said to be periodic if it satisfies the condition $x(n) = x(n + N)$ for all integers n.

The smallest value of N which satisfies the above condition is known as fundamental period.

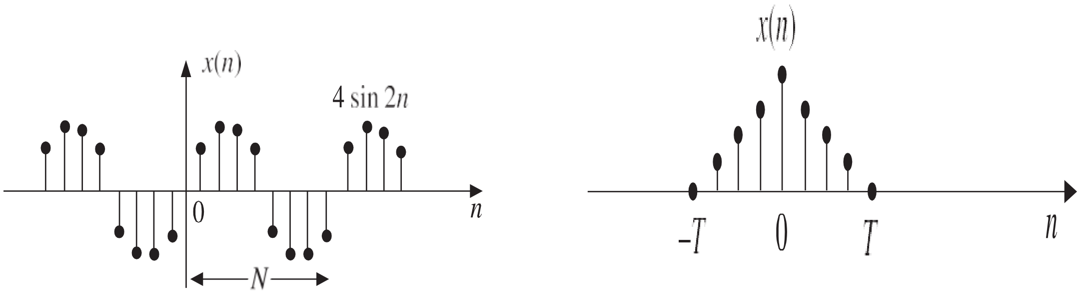
If the above condition is not satisfied even for one value of n, then the discrete-time signal is aperiodic. Sometimes aperiodic signals are said to have a period equal to infinity.

The angular frequency is given by

$$\omega = \frac{2\pi}{N}$$

Fundamental period $\quad N = \frac{2\pi}{\omega}$

The sum of two discrete-time periodic sequence is always periodic.



some examples of discrete-time periodic/non-periodic signals are shown in Figure 1.19.

Figure 1.19 Example of discrete-time: (a) Periodic and (b) Non-periodic signals

EXAMPLE 1.4 Show that the complex exponential sequence $x(n) = e^{j\ _0 n}$ is periodic only if $_0/2$ is a rational number.

Solution: Given $\quad x(n) = e^{j\omega_0 n}$

X (n) will be periodic if $\quad x(n + N) = x(n)$

i.e. $\quad e^{j[\omega_0(n=N0]} = e^{j\omega_0 n}$

i.e. $\quad e^{j\omega_0 N}\, e^{j\omega_0 n} = e^{j\omega_0 n}$

This is possible only if $\quad e^{j\ _0 N} = 1$

This is true only if $\quad \omega_0 N = 2\pi k$

Where k is an integer $\quad \frac{\omega_0}{2\pi} = \frac{k}{N}$

1.5.3 Energy Signals And Power Signals

Signals may also be classified as energy signals and power signals. However there are some signals which can neither be classified as energy signals nor power signals.

The total energy E of a discrete-time signal $x(n)$ is defined as:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

and the average power P of a discrete-time signal $x(n)$ is defined as:

$$P = \underset{N\to\infty}{\mathrm{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

or $P = \frac{1}{N}\sum_{n=0}^{n-1} |x(n)|^2$ for a digital signal with $x(n) = 0$ for $n < 0$.

A signal is said to be an energy signal if and only if its total energy E over the interval $(-\infty, \infty)$ is finite (i.e., $0 < E < \infty$). For an energy signal, average power $P = 0$. Non-periodic signals which are defined over a finite time (also called time limited signals) are the examples of energy signals. Since the energy of a periodic signal is always either zero or infinite, any periodic signal cannot be an energy signal.

A signal is said to be a power signal, if its average power P is finite (i.e., $0 < P < \infty$). For a power signal, total energy $E = \infty$. Periodic signals are the examples of power signals. Every bounded and periodic signal is a power signal. But it is true that a power signal is not necessarily a bounded and periodic signal.

Both energy and power signals are mutually exclusive, i.e. no signal can be both energy signal and power signal.

The signals that do not satisfy the above properties are neither energy signals nor power signals. For example, $x(n) = u(n)$, $x(n) = nu(n)$, $x(n) = n^2u(n)$.

These are signals for which neither P nor E are finite. If the signals contain infinite energy and zero power or infinite energy and infinite power, they are neither energy nor power signals.

If the signal amplitude becomes zero as $|n| \to \infty$, it is an energy signal, and if the signal amplitude does not become zero as $|n| \to \infty$, it is a power signal.

Causal and Non-causal Signals

A discrete-time signal $x(n)$ is said to be causal if $x(n) = 0$ for $n < 0$, otherwise the signal is non-causal. A discrete-time signal $x(n)$ is said to be anti-causal if $x(n) = 0$ for $n > 0$.

A causal signal does not exist for negative time and an anti-causal signal does not exist for positive time. A signal which exists in positive as well as negative time is called a non-casual signal.

$u(n)$ is a causal signal and $u(-n)$ an anti-causal signal, whereas $x(n) = 1$ for $-2 \le n \le 3$ is a non-causal signal.

Even and Odd Signals

Any signal $x(n)$ can be expressed as sum of even and odd components. That is

$$x(n) = x_e(n) + x_o(n)$$

where $x_e(n)$ is even components and $x_o(n)$ is odd components of the signal.

Even (syMMetric) signal

A discrete-time signal $x(n)$ is said to be an even (symmetric) signal if it satisfies the condition:

$$x(n) = x(-n) \quad \text{for all } n$$

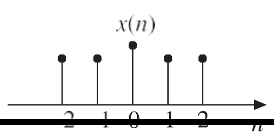
Even signals are symmetrical about the vertical axis or time origin. Hence they are also called symmetric signals: cosine sequence is an example of an even signal. Some even signals are shown in Figure 1.20(a). An even signal is identical to its reflection about the origin. For an even signal $x_0(n) = 0$.

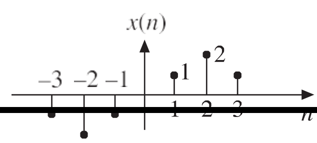
Odd (anti-syMMetric) signal

A discrete-time signal $x(n)$ is said to be an odd (anti-symmetric) signal if it satisfies the condition:

$$x(-n) = -x(n) \quad \text{for all } n$$

Odd signals are anti-symmetrical about the vertical axis. Hence they are called anti- symmetric signals. Sinusoidal sequence is an example of an odd signal. For an odd signal $x_e(n) = 0$. Some odd signals are shown in Figure 1.20(b).





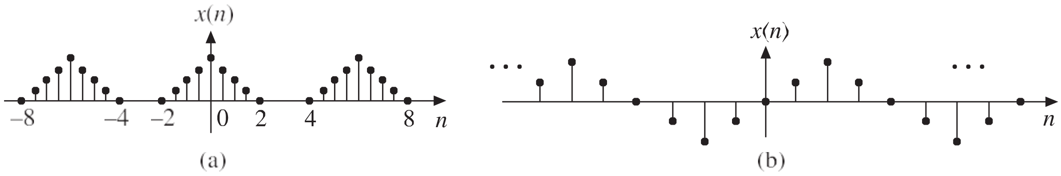


Figure 1.20 (a) Even sequences (b) Odd sequences.

Thus, the product of two even signals or of two odd signals is an even signal, and the product of even and odd signals is an odd signal.

Every signal need not be either purely even signal or purely odd signal, but every signal can be decomposed into sum of even and odd parts.

CLASSIFICATIOK OF DISCRETE-TImE SYSTEmS

A system is defined as an entity that acts on an input signal and transforms it into an output signal. A system may also be defined as a set of elements or functional blocks which are connected together and produces an output in response to an input signal. The response or output of the system depends on the transfer function of the system. It is a cause and effect relation between two or more signals.

As signals, systems are also broadly classified into continuous-time and discrete-time systems. A continuous-time system is one which transforms continuous-time input signals into continuous-time output signals, whereas a discrete-time system is one which transforms discrete-time input signals into discrete-time output signals.

For example microprocessors, semiconductor memories, shift registers, etc. are discrete-time systems.

A discrete-time system is represented by a block diagram as shown in Figure 1.22. An arrow entering the box is the input signal (also called excitation, source or driving function) and an arrow leaving the box is an output signal (also called response). Generally, the input is denoted by $x(n)$ and the output is denoted by $y(n)$.

The relation between the input $x(n)$ and the output $y(n)$ of a system has the form:

$$y(n) = \text{Operation on } x(n)$$

Mathematically,

$$y(n) = T[x(n)]$$

which represents that $x(n)$ is transformed to $y(n)$. In other words, $y(n)$ is the transformed version of $x(n)$.

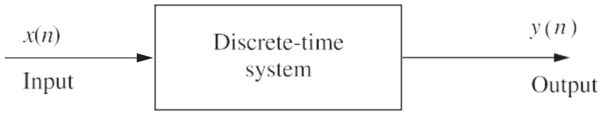


Figure 1.22 Block diagram of discrete-time system.

Both continuous-time and discrete-time systems are further classified as follows:

1. Static (memoryless) and dynamic (memory) systems
2. Causal and non-causal systems
3. Linear and non-linear systems
4. Time-invariant and time varying systems
5. Stable and unstable systems.
6. Invertible and non-invertible systems
7. FIR and IIR systems

Static and Dynamic Systems

A system is said to be static or memoryless if the response is due to present input alone, i.e., for a static or memoryless system, the output at any instant n depends only on the input applied at that instant n but not on the past or future values of input or past values of output.

For example, the systems defined below are static or memoryless systems.

$$y(n) = x(n)$$

$$y(n) = 2x^2(n)$$

In contrast, a system is said to be dynamic or memory system if the response depends upon past or future inputs or past outputs. A summer or accumulator, a delay element is a discrete-time system with memory.

For example, the systems defined below are dynamic or memory systems.

$$y(n) = x(2n)$$
$$y(n) = x(n) + x(n-2)$$
$$y(n) + 4y(n-1) + 4y(n-2) = x(n)$$

Any discrete-time system described by a difference equation is a dynamic system.

A purely resistive electrical circuit is a static system, whereas an electric circuit having inductors and/or capacitors is a dynamic system.

A discrete-time LTI system is memoryless (static) if its impulse response $h(n)$ is zero for n s 0. If the impulse response is not identically zero for n s 0, then the system is called dynamic system or system with memory.

EXAMPLE 1.12 Find whether the following systems are dynamic or not:

(a) $y(n) = x(n + 2)$ (b) $y(n) = x^2(n)$

(c) $y(n) = x(n - 2) + x(n)$

Solution:

(a) Given $y(n) = x(n + 2)$

The output depends on the future value of input. Therefore, the system is dynamic.

(b) Given $y(n) = x^2(n)$

The output depends on the present value of input alone. Therefore, the system is static.

(c) Given $y(n) = x(n - 2) + x(n)$

The system is described by a difference equation. Therefore, the system is dynamic.

Causal and Non-causal Systems

A system is said to be causal (or non-anticipative) if the output of the system at any instant n depends only on the present and past values of the input but not on future inputs, i.e., for a causal system, the impulse response or output does not begin before the input function is applied, i.e., a causal system is non anticipatory.

Causal systems are real time systems. They are physically realizable.

The impulse response of a causal system is zero for $n < 0$, since (n) exists only at $n = 0$,

i.e. $h(n) = 0 \quad \text{for} \quad n < 0$

The examples for causal systems are:

$$y(n) = nx(n)$$

$$y(n) = x(n - 2) + x(n - 1) + x(n)$$

A system is said to be non-causal (anticipative) if the output of the system at any instant n depends on future inputs. They are anticipatory systems. They produce an output even before the input is given. They do not exist in real time. They are not physically realizable.

A delay element is a causal system, whereas an image processing system is a non-causal system.

The examples for non-causal systems are:

$$y(n) = x(n) + x(2n)$$

$$y(n) = x^2(n) + 2x(n + 2)$$

EXAMPLE 1.13 Check whether the following systems are causal or not:

(a) $y(n)\ x(n)\ x(n\ 2)$ (b) $y(n) = x(2n)$

(c) $y(n) = \sin[x(n)]$ (d) $y(n) = x(-n)$

Solution:

(a) Given $y(n)\ x(n)\ x(n\ 2)$

For $n = -2$ $y(\ 2)\ x(\ 2)\ x(\ 4)$

For $n = 0$ $y(0)\ x(0)\ x(\ 2)$

For $n = 2$ $y(2)\ x(2)\ x(0)$

For all values of n, the output depends only on the present and past inputs. Therefore, the system is causal.

(a) Given $y(n)\quad x(2n)$

For $n = -2$ $y(\ 2)\quad x(\ 4)$

For $n = 0$ $y(0)\quad x(0)$

For $n = 2$ $y(2)\quad x(4)$

For positive values of n, the output depends on the future values of input. Therefore, the system is non-causal.

(a) Given $y(n)\ \sin[x(n)]$

For $n = -2$ $y(\ 2)\ \sin[x(\ 2)]$

For $n = 0$ $y(0)\quad \sin[x(0)]$

For $n = 2$ $y(2)\quad \sin[x(2)]$

For all values of n, the output depends only on the present value of input. Therefore, the system is causal.

(d) Given $y(n) = x(-n)$

For $n = -2$ $y(\ 2)\ x(2)$

For $n = 0$ $y(0)\ x(0)$

For $n = 2$ $y(2)\ x(\ 2)$

For negative values of n, the output depends on the future values of input. Therefore, the system is non-causal.

Linear and Non-linear Systems

A system which obeys the principle of superposition and principle of homogeneity is called a linear system and a system which does not obey the principle of superposition and homogeneity is called a non-linear system.

Homogeneity property means a system which produces an output $y(n)$ for an input $x(n)$ must produce an output $ay(n)$ for an input $ax(n)$.

Superposition property means a system which produces an output $y_1(n)$ for an input $x_1(n)$ and an output $y_2(n)$ for an input $x_2(n)$ must produce an output $y_1(n) + y_2(n)$ for an input $x_1(n) + x_2(n)$.

Combining them we can say that a system is linear if an arbitrary input $x_1(n)$ produces an output $y_1(n)$ and an arbitrary input $x_2(n)$ produces an output $y_2(n)$, then the weighted sum of inputs $ax_1(n) + bx_2(n)$ where a and b are constants produces an output $ay_1(n) + by_2(n)$ which is the sum of weighted outputs.

$$T(ax_1(n) + bx_2(n)] = aT[x_1(n)] + bT[x_2(n)]$$

Simply we can say that a system is linear if the output due to weighted sum of inputs is equal to the weighted sum of outputs.

In general, if the describing equation contains square or higher order terms of input and/or output and/or product of input/output and its difference or a constant, the system will definitely be non-linear.

Shift-invariant and Shift-varying Systems

Time-invariance is the property of a system which makes the behaviour of the system independent of time. This means that the behaviour of the system does not depend on the time at which the input is applied. For discrete-time systems, the time invariance property is called shift invariance.

A system is said to be shift-invariant if its input/output characteristics do not change with time, i.e., if a time shift in the input results in a corresponding time shift in the output as shown in Figure 1.23, i.e.

If $T[x(n)] = y(n)$

Then $T[x(n - k)] = y(n - k)$

A system not satisfying the above requirements is called a time-varying system (or shift-varying system). A time-invariant system is also called a fixed system.

The time-invariance property of the given discrete-time system can be tested as follows:

Let $x(n)$ be the input and let $x(n - k)$ be the input delayed by k units.

$y(n) = T[x(n)]$ be the output for the input $x(n)$.

Stable and Unstable Systems

A bounded signal is a signal whose magnitude is always a finite value, i.e. $|x(n)| \leq M$, where M is a positive real finite number. For example a sinewave is a bounded signal. A system is said to be bounded-input, bounded-output (BIBO) stable, if and only if every bounded input produces a bounded output. The output of such a system does not diverge or does not grow unreasonably large.

Let the input signal $x(n)$ be bounded (finite), i.e.,

$$|x(n)| \leq M_x \qquad \text{for all } n$$

where M_x is a positive real number. If

$$|y(n)| \leq M_y \leq \infty$$

i.e. if the output $y(n)$ is also bounded, then the system is BIBO stable. Otherwise, the system is unstable. That is, we say that a system is unstable even if one bounded input produces an unbounded output.

It is very important to know about the stability of the system. Stability indicates the usefulness of the system. The stability can be found from the impulse response of the system which is nothing but the output of the system for a unit impulse input. If the impulse response is absolutely summable for a discrete-time system, then the system is stable.

BIBO stability criterion

The necessary and sufficient condition for a discrete-time system to be BIBO stable is given by the expression:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

where $h(n)$ is the impulse response of the system. This is called BIBO stability criterion.

Proof: Consider a linear time-invariant system with $x(n)$ as input and $y(n)$ as output. The input and output of the system are related by the convolution integral.