As a result, an LTI discrete-time system is *causal* if and only if its impulse response sequence $\{h[n]\}$ is a causal sequence satisfying the condition of Eq. (2.80).

It follows from Example 2.21 that the discrete-time system of Eq. (2.14) is a causal system since its impulse response satisfies the causality condition of Eq. (2.80). Likewise, from Example 2.22 we observe that the discrete-time accumulator of Eq. (2.54) is also a causal system. On the other hand, from Example 2.23 it can be seen that the factor-of-2 linear interpolator defined by Eq. (2.58) is a noncausal system because its impulse response does not satisfy the causality condition of Eq. (2.80). However, a noncausal discrete-time system with a finite-length impulse response can often be realized as a causal system by inserting a delay of an appropriate amount. For example, a causal version of the discrete-time factor-of-2 linear interpolator is obtained by delaying the output by one sample period with an input-output relation given by

$$y[n] = x_u[n-1] + \frac{1}{2}(x_u[n-2] + x_u[n]).$$

2.6 Finite-Dimensional LTI Discrete-Time Systems

An important subclass of LTI discrete-time systems is characterized by a linear constant coefficient difference equation of the form

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k], \quad (2.81)$$

where $x[n]$ and $y[n]$ are, respectively, the input and the output of the system, and $\{d_k\}$ and $\{p_k\}$ are constants. The *order* of the discrete-time system is given by $\max(N, M)$, which is the order of the difference equation characterizing the system. It is possible to implement an LTI system characterized by Eq. (2.81) since the computation here involves two finite sum of products even though such a system, in general, has an impulse response of infinite length.

The output $y[n]$ can then be computed recursively from Eq. (2.81). If we assume the system to be causal, then we can rewrite Eq. (2.81) to express $y[n]$ explicitly as a function of $x[n]$:

$$y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k], \quad (2.82)$$

provided $d_0 \neq 0$. The output $y[n]$ can be computed for all $n \geq n_o$ knowing $x[n]$ and the initial conditions $y[n_o - 1], y[n_o - 2], \ldots, y[n_o - N]$.

2.6.1 Total Solution Calculation

The procedure for computing the solution of the constant coefficient difference equation of Eq. (2.81) is very similar to that employed in solving the constant coefficient differential equation in the case of an LTI continuous-time system. In the case of the discrete-time system of Eq. (2.81) the output response $y[n]$ also consists of two components which are computed independently and then added to yield the total solution:

$$y[n] = y_c[n] + y_p[n]. \quad (2.83)$$

In the above equation the component $y_c[n]$ is the solution of Eq. (2.81) with the input $x[n] = 0$; i.e., it is the solution of the homogeneous difference equation:

$$\sum_{k=0}^{N} d_k y[n-k] = 0. \quad (2.84)$$

and the component $y_p[n]$ is a solution of Eq. (2.81) with $x[n] \neq 0$. $y_c[n]$ is called the *complementary solution*, while $y_p[n]$ is called the *particular solution* resulting from the specified input $x[n]$, often called the *forcing function*. The sum of the complementary and the particular solutions as given by Eq. (2.83) is called the *total solution*.

We first describe the method of computing the complementary solution $y_c[n]$. To this end we assume that it is of the form

$$y_c[n] = \lambda^n. \tag{2.85}$$

Substituting the above in Eq. (2.84) we arrive at

$$\begin{aligned}\sum_{k=0}^{N} d_k y[n-k] &= \sum_{k=0}^{N} d_k \lambda^{n-k} \\ &= \lambda^{n-N}(d_0\lambda^N + d_1\lambda^{N-1} + \cdots + d_{N-1}\lambda + d_N) = 0.\end{aligned} \tag{2.86}$$

The polynomial $\sum_{k=0}^{N} d_k \lambda^{N-k}$ is called the *characteristic polynomial* of the discrete-time system of Eq. (2.81). Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ denote its N roots. If these roots are all distinct, then the general form of the complementary solution is given by

$$y_c[n] = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_N\lambda_N^n, \tag{2.87}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_N$ are constants determined from the specified initial conditions of the discrete-time system. The complementary solution takes a different form in the case of multiple roots. For example, if λ_1 is of multiplicity L and the remaining $N-L$ roots, $\lambda_2, \lambda_3, \ldots, \lambda_{N-L}$, are distinct, then Eq. (2.87) takes the form

$$y_c[n] = \alpha_1\lambda_1^n + \alpha_2 n\lambda_1^n + \alpha_3 n^2\lambda_1^n + \cdots + \alpha_L n^{L-1}\lambda_1^n + \alpha_{L+1}\lambda_2^n + \cdots + \alpha_N\lambda_{N-L}^n. \tag{2.88}$$

Next, we consider the determination of the particular solution $y_p[n]$ of the difference equation of Eq. (2.81). Here the procedure is to assume that the particular solution is also of the same form as the specified input $x[n]$ if $x[n]$ has the form $\lambda_0^n (\lambda_0 \neq \lambda_i, i = 1, 2, \ldots, N)$ for all n. Thus, if $x[n]$ is a constant, then $y_p[n]$ is also assumed to be constant. Likewise, if $x[n]$ is a sinusoidal sequence, then $y_p[n]$ is also assumed to be a sinusoidal sequence, and so on.

We illustrate below the determination of the total solution by means of an example.

EXAMPLE 2.30 Let us determine the total solution for $n \geq 0$ of a discrete-time system characterized by the following difference equation:

$$y[n] + y[n-1] - 6y[n-2] = x[n], \tag{2.89}$$

for a step input $x[n] = 8\mu[n]$ and with initial conditions $y[-1] = 1$ and $y[-2] = -1$.

We first determine the form of the complementary solution. Setting $x[n] = 0$ and $y[n] = \lambda^n$ in Eq. (2.89) we arrive at

$$\begin{aligned}\lambda^n + \lambda^{n-1} - 6\lambda^{n-2} &= \lambda^{n-2}(\lambda^2 + \lambda - 6) \\ &= \lambda^{n-2}(\lambda+3)(\lambda-2) = 0,\end{aligned}$$

and hence the roots of the characteristic polynomial $\lambda^2 + \lambda - 6$ are $\lambda_1 = -3, \lambda_2 = 2$. Therefore the complementary solution is of the form

$$y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n. \tag{2.90}$$

For the particular solution we assume

$$y_p[n] = \beta.$$

Substituting the above in Eq. (2.89) we get

$$\beta + \beta - 6\beta = 8\mu[n],$$

which for $n \geq 0$ yields $\beta = -2$.

The total solution is therefore of the form

$$y[n] = \alpha_1(-3)^n + \alpha_2(2)^n - 2, \qquad n \geq 0. \tag{2.91}$$

The constants α_1 and α_2 are chosen to satisfy the specified initial conditions. From Eqs. (2.89) and (2.91) we get

$$y[-2] = \alpha_1(-3)^{-2} + \alpha_2(2)^{-2} - 2 = -1,$$
$$y[-1] = \alpha_1(-3)^{-1} + \alpha_2(2)^{-1} - 2 = 1.$$

Solving these two equations we arrive at

$$\alpha_1 = -1.8, \ \alpha_2 = 4.8.$$

Thus, the total solution is given by

$$y[n] = -1.8(-3)^n + 4.8(2)^n - 2, \qquad n \geq 0.$$

If the input excitation is of the same form as one of the terms in the complementary solution, then it is necessary to modify the form of the particular solution as illustrated in the following example.

EXAMPLE 2.31 We determine the total solution for $n \geq 0$ of the difference equation of Eq. (2.89) for an input $x[n] = (2)^n \mu[n]$ with the same initial conditions as in the previous example.

As indicated in the previous example, the complementary solution contains a term $\alpha_2(2)^n$, which is of the same form as the specified input. Hence we need to select a form for the particular solution which is distinct and does not contain any terms similar to those contained in the complementary solution. We assume

$$y_p[n] = \beta n(2)^n.$$

Substituting the above in Eq. (2.89) we get

$$\beta n(2)^n + \beta(n-1)(2)^{n-1} - 6\beta(n-2)(2)^{n-2} = (2)^n \mu[n].$$

For $n \geq 0$ we obtain from the above equation $\beta = 0.4$. The total solution is now of the form

$$y[n] = \alpha_1(-3)^n + \alpha_2(2)^n + 0.4n(2)^n, \qquad n \geq 0. \tag{2.92}$$

To determine the values of α_1 and α_2, we make use of the specified initial conditions. From Eqs. (2.89) and (2.92) we arrive at

$$y[-2] = \alpha_1(-3)^{-2} + \alpha_2(2)^{-2} + 0.4(-2)(2)^{-2} = -1,$$
$$y[-1] = \alpha_1(-3)^{-1} + \alpha_2(2)^{-1} + 0.4(-1)(2)^{-1} = 1,$$

which when solved yields $\alpha_1 = -5.04$, $\alpha_2 = -0.96$. Therefore the total solution is given by

$$y[n] = -5.04(-3)^n - 0.96(2)^n + 0.4n(2)^n, \qquad n \geq 0.$$

2.6.2 Zero-Input Response and Zero-State Response

An alternate approach to determining the total solution $y[n]$ of the difference equation of Eq. (2.81) is by computing its *zero-input response* $y_{zi}[n]$ and *zero-state response* $y_{zs}[n]$. The component $y_{zi}[n]$ is obtained by solving Eq. (2.81) by setting the input $x[n] = 0$, and the component $y_{zs}[n]$ is obtained by solving Eq. (2.81) by applying the specified input with all initial conditions set to zero. The total solution is then given by $y_{zi}[n] + y_{zs}[n]$.

This approach is illustrated by the following example.

EXAMPLE 2.32 We determine the total solution of the discrete-time system of Example 2.30 by computing the zero-input response and the zero-state response.

The zero-input response $y_{zi}[n]$ of Eq. (2.89) is given by the complementary solution of Eq. (2.90) where the constants α_1 and α_2 are chosen to satisfy the specified initial conditions. Now from Eq. (2.89) we get

$$y[0] = -y[-1] + 6y[-2] = -1 - 6 = -7,$$
$$y[1] = -y[0] + 6y[-1] = 7 + 6 = 13.$$

Next from Eq. (2.90) we get

$$y[0] = \alpha_1 + \alpha_2,$$
$$y[1] = -3\alpha_1 + 2\alpha_2.$$

Solving these two sets of equations we arrive at $\alpha_1 = -5.4$, $\alpha_2 = -1.6$. Therefore

$$y_{zi}[n] = -5.4(-3)^n - 1.6(2)^n, \qquad n \geq 0.$$

The zero-state response is determined from Eq. (2.91) by evaluating the constants α_1 and α_2 to satisfy the zero initial conditions. From Eq. (2.89) we get

$$y[0] = x[0] = 8,$$
$$y[1] = x[1] - y[0] = 0.$$

Next, from Eq. (2.91) and the above set of equations we arrive at $\alpha_1 = 3.6$, $\alpha_2 = 6.4$. Thus, the zero-state response for $n \geq 0$ with initial conditions $y_{zs}[-2] = y_{zs}[-1] = 0$ is given by

$$y_{zs}[n] = 3.6(-3)^n + 6.4(2)^n - 2.$$

Hence, the total solution $y[n]$ is given by the sum $y_{zs}[n] + y_{zi}[n]$ resulting in

$$y[n] = -1.8(-3)^n + 4.8(2)^n - 2, \qquad n \geq 0,$$

which is identical to that derived in Example 2.30 as expected.

2.6.3 Impulse Response Calculation

The impulse response $h[n]$ of a causal LTI discrete-time system is the output observed with input $x[n] = \delta[n]$. Thus, it is simply the zero-state response with $x[n] = \delta[n]$. Now for such an input, $x[n] = 0$ for $n > 0$, and thus, the particular solution is zero, i.e., $y_p[n] = 0$. Hence the impulse response can be computed from the complementary solution of Eq. (2.87) in the case of simple roots of the characteristic equation by determining the constants α_i to satisfy the zero initial conditions. A similar procedure can be followed in the case of multiple roots of the characteristic equation. A system with all zero initial conditions is often called a *relaxed* system.

EXAMPLE 2.33 In this example we determine the impulse response $h[n]$ of the causal discrete-time system of Example 2.30. From Eq. (2.90) we get

$$h[n] = \alpha_1(-3)^n + \alpha_2(2)^n, \qquad n \geq 0.$$

From the above, we arrive at

$$h[0] = \alpha_1 + \alpha_2,$$
$$h[1] = -3\alpha_1 + 2\alpha_2.$$

Next, from Eq. (2.89) with $x[n] = \delta[n]$ we get

$$h[0] = 1,$$
$$h[1] + h[0] = 0.$$

Solution of the above two sets of equations yields $\alpha_1 = 0.6$, $\alpha_2 = 0.4$.

Thus, the impulse response is given by

$$h[n] = 0.6(-3)^n + 0.4(2)^n, \qquad n \geq 0.$$

It follows from the form of the complementary solution given by Eq. (2.88) that the impulse response of a finite-dimensional LTI system characterized by a difference equation of the form of Eq. (2.81) is of infinite length. However, as illustrated by the following example, there exist infinite impulse response LTI discrete-time systems that cannot be characterized by the difference equation form of Eq. (2.81).

EXAMPLE 2.34 The system defined by the impulse response

$$h[n] = \frac{1}{n^2}\mu[n-1]$$

does not have a representation in the form of a linear constant coefficient difference equation. It should be noted that the above system is causal and also BIBO stable.

Since the impulse response $h[n]$ of a causal discrete-time system is a causal sequence, Eq. (2.82) can also be used to calculate recursively the impulse response for $n \geq 0$ by setting initial conditions to zero values, i.e., by setting $y[-1] = y[-2] = \cdots = y[-N] = 0$, and using a unit sample sequence $\delta[n]$ as the input $x[n]$. The step response of a causal LTI system can similarly be computed recursively by setting zero initial conditions and applying a unit step sequence as the input. It should be noted that the causal discrete-time system of Eq. (2.82) is linear only for zero initial conditions (Problem 2.45).

2.6.4 Output Computation Using MATLAB

The causal LTI system of the form of Eq. (2.82) can be simulated in MATLAB using the function `filter` already made use of in Program 2_4. In one of its forms, the function

```
y = filter(p,d,x)
```

processes the input data vector `x` using the system characterized by the coefficient vectors `p` and `d` to generate the output vector `y` assuming zero initial conditions. The length of `y` is the same as the length of `x`. Since the function implements Eq. (2.82), the coefficient d_0 must be nonzero.

The following example illustrates the computation of the impulse and step responses of an LTI system described by Eq. (2.82).