(c) Given $x(n) = a^n$

$$X(k) = \sum_{n=0}^{N-1} a^n e^{-j(2\pi/N)nk} = \sum_{n=0}^{N-1} [ae^{-j(2\pi/N)k}]^n \text{ for } 0 \leq k \leq N-1$$

$$= \frac{1 - a^N e^{-j2\pi k}}{1 - ae^{-j(2\pi/N)k}}$$

(d) Given $x(n) = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)nk}$$

$$= \sum_{n=0}^{(N/2)-1} x(2n) e^{-j(2\pi/N)2nk} + \sum_{n=0}^{(N/2)-1} x(2n+1) e^{-j(2\pi/N)(2n+1)k}$$

$$= \sum_{n=0}^{(N/2)-1} x(2n) e^{-j(4\pi/N)nk} = \sum_{n=0}^{(N/2)-1} e^{-j4\pi kn/N}$$

EXAMPLE (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.

(b) Find the IDFT of $X(k) = \{4, 2, 0, 4\}$ directly.

Solution:

(a) Given sequence is $x(n) = \{1, -1, 2, -2\}$. Here the DFT $X(k)$ to be found is $N =$ 4-point and length of the sequence $L = 4$. So no padding of zeros is required.

We know that the DFT $\{x(n)\}$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)nk} = \sum_{n=0}^{3} x(n) e^{-j(\pi/2)nk}, \quad k = 0, 1, 2, 3$$

$$\therefore \quad X(0) = \sum_{n=0}^{3} x(n)\, e^0 = x(0) + x(1) + x(2) + x(3) = 1 - 1 + 2 - 2 = 0$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j(\pi/2)n} = x(0) + x(1) e^{-(j\pi/2)} + x(2) e^{-j\pi} + x(3) e^{-j(3\pi/2)}$$

$$= 1 + (-1)(0 - j) + 2(-1 - j0) - 2(0 + j)$$

$$= -1 - j$$

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{-j\pi n} = x(0) + x(1)\, e^{-j\pi} + x(2)\, e^{-j2\pi} + x(3)\, e^{-j3\pi}$$

$$= 1 - 1(-1 - j0) + 2(1 - j0) - 2(-1 - j0) = 6$$

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{-j(3\pi/2)n} = x(0) + x(1)\, e^{-j(3\pi/2)} + x(2)\, e^{-j3\pi} + x(3)\, e^{-j(9\pi/2)}$$

$$= 1 - 1(0 + j) + 2(-1 - j0) - 2(0 - j) = -1 + j$$

$\therefore$ $X(k) = \{0,\ -1 - j,\ 6,\ -1 + j\}$

(b) Given DFT is $X(k) = \{4, 2, 0, 4\}$. The IDFT of $X(k)$, i.e. $x(n)$ is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-nk} = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j(2\pi/N)nk}$$

i.e. $$x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j(\pi/2)nk}$$

$\therefore$ $$x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{0} = \frac{1}{4}[X(0) + X(1) + X(2) + X(3)]$$

$$= \frac{1}{4}[4 + 2 + 0 + 4] = 2.5$$

$$x(1) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{j(\pi/2)k} = \frac{1}{4}\left[X(0) + X(1)\, e^{j(\pi/2)} + X(2)\, e^{j\pi} + X(3) e^{j(3\pi/2)}\right]$$

$$= \frac{1}{4}[4 + 2(0 + j) + 0 + 4(0 - j)] = 1 - j0.5$$

$$x(2) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{j\pi k} = \frac{1}{4}\left[X(0) + X(1)\, e^{j\pi} + X(2)\, e^{j2\pi} + X(3) e^{j3\pi}\right]$$

$$= \frac{1}{4}[4 + 2(-1 + j0) + 0 + 4(-1 + j0)] = -0.5$$

$$x(3) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{j(3\pi/2)k} = \frac{1}{4}\left[X(0) + X(1)\, e^{j(3\pi/2)} + X(2)\, e^{j3\pi} + X(3) e^{j(9\pi/2)}\right]$$

$$= \frac{1}{4}[4 + 2(0 - j) + 0 + 4(0 + j)] = 1 + j0.5$$

$$x_3(n) = \{2.5,\ 1 - j0.5,\ -0.5,\ 1 + j0.5\}$$

EXAMPLE (a) Find the 4-point DFT of $x(n) = \{1, -2, 3, 2\}$.
(b) Find the IDFT of $X(k) = \{1, 0, 1, 0\}$.

Solution:

(a) Given $x(n) = \{1, -2, 3, 2\}$.

Here $N = 4$, $L = 4$. The DFT of $x(n)$ is $X(k)$.

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{3} x(n) e^{-j(2\pi/4)nk} = \sum_{n=0}^{3} x(n) e^{-j(\pi/2)nk}, \quad k = 0, 1, 2, 3$$

$$X(0) = \sum_{n=0}^{3} x(n) e^{0} = x(0) + x(1) + x(2) + x(3) = 1 - 2 + 3 + 2 = 4$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j(\pi/2)n} = x(0) + x(1) e^{-j(\pi/2)} + x(2) e^{-j\pi} + x(3) e^{-j(3\pi/2)}$$

$$= 1 - 2(0 - j) + 3(-1 - j0) + 2(0 + j) = -2 + j4$$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n} = x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$$

$$= 1 - 2(-1 - j0) + 3(1 - j0) + 2(-1 - j0) = 4$$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j(3\pi/2)n} = x(0) + x(1) e^{-j(3\pi/2)} + x(2) e^{-j3\pi} + x(3) e^{-j(9\pi/2)}$$

$$= 1 - 2(0 + j) + 3(-1 - j0) + 2(0 - j) = -2 - j4$$

$$\therefore \quad X(k) = \{4, -2 + j4, 4, -2 - j4\}$$

(b) Given $X(k) = \{1, 0, 1, 0\}$

Let the IDFT of $X(k)$ be $x(n)$.

$$\therefore \quad x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,W_N^{-nk} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,e^{j(2\pi/N)nk}$$

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{0} = \frac{1}{4}[X(0)+X(1)+X(2)+X(3)] = \frac{1}{4}[1+0+1+0] = 0.5$$

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j(\pi/2)k} = \frac{1}{4}\left[X(0)+X(1)e^{j(\pi/2)}+X(2)e^{j\pi}+X(3)e^{j(3\pi/2)}\right]$$

$$= \frac{1}{4}[1+0+e^{j\pi}+0] = \frac{1}{4}[1+0-1+0] = 0$$

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\pi k} = \frac{1}{4}\left[X(0)+X(1)e^{j\pi}+X(2)e^{j2\pi}+X(3)e^{j3\pi}\right]$$

$$= \frac{1}{4}[1+0+e^{j2\pi}+0] = \frac{1}{4}[1+0+1+0] = 0.5$$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j(3\pi/2)k} = \frac{1}{4}\left[X(0)+X(1)e^{j(3\pi/2)}+X(2)e^{j3\pi}+X(3)e^{j(9\pi/2)}\right]$$

$$= \frac{1}{4}[1+0+e^{j3\pi}+0] = \frac{1}{4}[1+0-1+0] = 0$$

$\therefore$ The IDFT of $X(k) = \{1, 0, 1, 0\}$ is $x(n) = \{0.5, 0, 0.5, 0\}$.

EXAMPLE Compute the DFT of the 3-point sequence $x(n) = \{2, 1, 2\}$. Using the same sequence, compute the 6-point DFT and compare the two DFTs.

Solution: The given 3-point sequence is $x(n) = \{2, 1, 2\}$, $N = 3$.

$$\text{DFT } x(n) = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{2} x(n) e^{-j(2\pi/3)nk}, \quad k = 0, 1, 2$$

$$= x(0) + x(1) e^{-j(2\pi/3)k} + x(2) e^{-j(4\pi/3)k}$$

$$= 2 + \left(\cos\frac{2\pi}{3}k - j\sin\frac{2\pi}{3}k \right) + 2\left(\cos\frac{4\pi}{3}k - j\sin\frac{4\pi}{3}k \right)$$

When $k = 0$, $X(k) = X(0) = 2 + 1 + 2 = 5$

When $k = 1$, $X(k) = X(1) = 2 + \left(\cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3} \right) + 2\left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3} \right)$

$$= 2 + (-0.5 - j0.866) + 2(-0.5 + j0.866)$$

$$= 0.5 + j0.866$$

When $k = 2$, $X(k) = X(2) = 2 + \left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3} \right) + 2\left(\cos\frac{8\pi}{3} - j\sin\frac{8\pi}{3} \right)$

$$= 2 + (-0.5 + j0.866) + 2(-0.5 - j0.866)$$

$$= 0.5 - j0.866$$

$\therefore$ 3-point DFT of $x(n) = X(k) = \{5, 0.5 + j0.866, 0.5 - j0.866\}$

MATRIX FORMULATION OF THE DFT AND IDFT

If we let $W_N = e^{-j(2\pi/N)}$, the defining relations for the DFT and IDFT may be written as:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \qquad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk}, \qquad n = 0, 1, 2, \ldots, N-1$$

The first set of N DFT equations in N unknowns may be expressed in matrix form as:

$$\mathbf{X} = \mathbf{W_N x}$$

Here $\mathbf{X}$ and $\mathbf{x}$ are $N \times 1$ matrices, and $\mathbf{W_N}$ is an $N \times N$ square matrix called the DFT matrix. The full matrix form is described by

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

THE IDFT FROM THE MATRIX FORM

The matrix $\mathbf{x}$ may be expressed in terms of the inverse of $\mathbf{W_N}$ as:

$$\mathbf{x} = \mathbf{W_N^{-1}}\, \mathbf{x}$$

The matrix $\mathbf{W_N^{-1}}$ is called the IDFT matrix. We may also obtain $\mathbf{x}$ directly from the IDFT relation in matrix form, where the change of index from n to k and the change in the sign of the exponent in $e^{j(2\pi/N)nk}$ lead to the conjugate transpose of $\mathbf{W_N}$. We then have

$$\mathbf{x} = \frac{1}{\mathbf{N}}\left[\mathbf{W_N^*}\right]^T \mathbf{x}$$

Comparison of the two forms suggests that $\mathbf{W_N^{-1}} = \frac{1}{N}\left[\mathbf{W_N^*}\right]^T$.

This very important result shows that $\mathbf{W_N^{-1}}$ requires only conjugation and transposition of $\mathbf{W_N}$, an obvious computational advantage.

USING THE DFT TO FIND THE IDFT

Both the DFT and IDFT are matrix operations and there is an inherent symmetry in the DFT and IDFT relations. In fact, we can obtain the IDFT by finding the DFT of the conjugate sequence and then conjugating the results and dividing by N, Mathematically,

$$x(n) = \text{IDFT}[X(k)] = \frac{1}{N}[\text{DFT}\{X^*(k)\}]^*$$

This result involves the conjugate symmetry and duality of the DFT and IDFT, and suggests that the DFT algorithm itself can also be used to find the IDFT. In practice, this is indeed what is done.

EXAMPLE Find the DFT of the sequence

$$x(n) = \{1, 2, 1, 0\}$$

Solution: The DFT $X(k)$ of the given sequence $x(n) = \{1, 2, 1, 0\}$ may be obtained by solving the matrix product as follows. Here $N = 4$.

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & W_N^3 \\ W_N^0 & W_N^2 & W_N^4 & W_N^6 \\ W_N^0 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

The result is DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$.

EXAMPLE Find the DFT of $x(n) = \{1, -1, 2, -2\}$.

Solution: The DFT, $X(k)$ of the given sequence $x(n) = \{1, -1, 2, -2\}$ can be determined using matrix as shown below.

$$X(k) = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1-j \\ 6 \\ -1+j \end{bmatrix}$$

$\therefore$ DFT $\{x(n)\} = X(k) = \{0, -1-j, 6, -1+j\}$

EXAMPLE Find the 4-point DFT of $x(n) = \{1, -2, 3, 2\}$.

Solution: Given $x(n) = \{1, -2, 3, 2\}$, the 4-point DFT$\{x(n)\} = X(k)$ is determined using matrix as shown below.

$$\text{DFT}\{x(n)\} = X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2+j4 \\ 4 \\ -2-j4 \end{bmatrix}$$

$$\therefore \quad \text{DFT}\{x(n)\} = X(k) = \{4, -2+j4, 4, -2-j4\}$$

EXAMPLE Find the IDFT of $X(k)=\{4, -j2, 0, j2\}$ using DFT.

Solution: Given $X(k) = \{4, -j2, 0, j2\}$ $\therefore$ $X^*(k) = \{4, j2, 0, -j2\}$

The IDFT of $X(k)$ is determined using matrix as shown below.

To find IDFT of $X(k)$ first find $X^*(k)$, then find DFT of $X^*(k)$, then take conjugate of DFT $\{X^*(k)\}$ and divide by N.

$$\text{DFT}\{X^*(k)\} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ j2 \\ 0 \\ -j2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}[X(k)] = x(n) = \frac{1}{4}[4, 8, 4, 0]^* = \frac{1}{4}[4, 8, 4, 0] = [1, 2, 1, 0]$$

EXAMPLE Find the IDFT of $X(k) = \{4, 2, 0, 4\}$ using DFT.

Solution: Given $X(k) = \{4, 2, 0, 4\}$

$$\therefore \quad X^*(k) = \{4, 2, 0, 4\}$$

The IDFT of $X(k)$ is determined using matrix as shown below.

To find IDFT of $X(k)$, first find $X^*(k)$, then find DFT of $X^*(k)$, then take conjugate of DFT $\{X^*(k)\}$ and divide by N.

$$\text{DFT}[X^*(k)] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 4+j2 \\ -2 \\ 4-j2 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}\{X(k)\} = x(n) = \frac{1}{4}[10, 4+j2, -2, 4-j2]^* = \{2.5, 1-j0.5, -0.5, 1+j0.5\}$$

EXAMPLE Find the IDFT of $X(k) = \{1, 0, 1, 0\}$.

Solution: Given $X(k) = \{1, 0, 1, 0\}$, the IDFT of $X(k)$, i.e. $x(n)$ is determined using matrix as shown below.

$$X^*(k) = \{1, 0, 1, 0\}^* = \{1, 0, 1, 0\}$$

$$\text{DFT}\,\{X^*(k)\} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}\,\{X(k)\} = x(n) = \frac{1}{4}[\text{DFT}\,\{X^*(k)\}]^* = \frac{1}{4}\{2, 0, 2, 0\} = \{0.5, 0, 0.5, 0\}$$