Z-Transform of Finite duration signal

Find the Z - Transform and mention the Region of Convergence (ROC) for the following discrete time sequences.

1. $x(n) = \{\quad \underset{\uparrow}{2} \quad 1 \quad 2 \quad 3\}$
2. $x(n) = \{2, 1, 2\ 3\} \underset{\uparrow}{}$
3. $x(n) = \{\quad 1 \quad \underset{\uparrow}{} \quad 2\ 1 \quad -2\ 3\ 1\}$

1st example Is of causal signal

2nd example Is of anti-causal signal

3rd example Is of non-causal signal

Solution(1)

$x(n) = \{\underset{\uparrow}{2} \quad 1 \quad 2 \quad 3\}$

Z.T. is defined as $$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

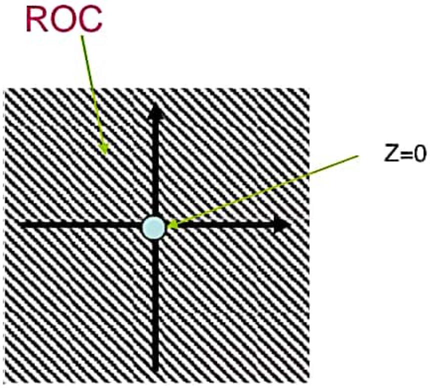
$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$X(z) = 2 + 1z^{-1} + 2z^{-2} + 3z^{-3}$$

$$X(z) = 2 + z^{-1} + 2z^{-2} + 3z^{-3}$$

ROC is a set of those values of z for which x (z) is not infinite

In this case x(z) is finite for all values of z, except |z| = 0.
Because at z = 0, x(z) = ∞.
Thus ROC is entire z-plane except |z| = 0.



Solution(2)

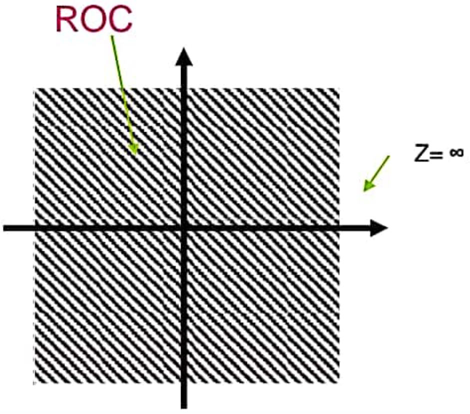
$x(n) = \{ \mathbf{2} \quad 1 \quad 2 \quad \underset{\uparrow}{3} \}$

Z.T. is defined as $$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = x(0) + x(-1)z^{1} + x(-2)z^{2} + x(-3)z^{3}$$

$$X(z) = 3 + 2z^{1} + 1z^{2} + 2z^{3}$$

$$X(z) = 3 + 2z + z^{2} + 2z^{3}$$



ROC is a set of those values of z for which x (z) is not infinite

In this case X(z) is finite for all values of z, except |z| = ∞.
Because at z = ∞, X(z) = ∞.
Thus ROC is entire z-plane except |z| = ∞.

Solution(3)

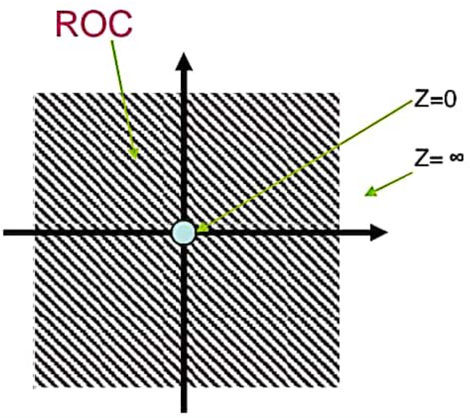
$x(n) = \{\ 1 \quad 2 \ \ \underset{\uparrow}{1}\ -2\ 3\ 1\}$

Z.T. is defined as $$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = x(-2)z^{2} + x(-1)z^{1} + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$X(z) = 1z^{2} + 2z^{1} + 1 - 2z^{-1} + 3z^{-2} + 1z^{-3}$$

$$X(z) = z^{2} + 2z^{1} + 1 - 2z^{-1} + 3z^{-2} + z^{-3}$$



ROC is a set of those values of z for which x (z) is not infinite

In this case X(z) is finite for all values of z, except |z| = ∞.
Because at z = ∞, X(z) = ∞.
Thus ROC is entire z-plane except |z| = 0 &|z| = ∞.

z-Transform of infinite duration signal

Find the z-transform for following discrete time sequences. Also mention ROC for all the cases.

1 $x(n) = a^n U(n)$

2 $x(n) = -a^n U(-n-1)$

3 $x(n) = a^n U(n) + b^n U(-n-1)$

Solution(1)

$x(n) = a^n U(n)$ Sequence is causal

Z.T. for the given sequence x (n) is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n U(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

We know that $\sum_{n=0}^{\infty} a^n = a^0 + a^1 + a^2 + ...$

Series converges iff |a|<1

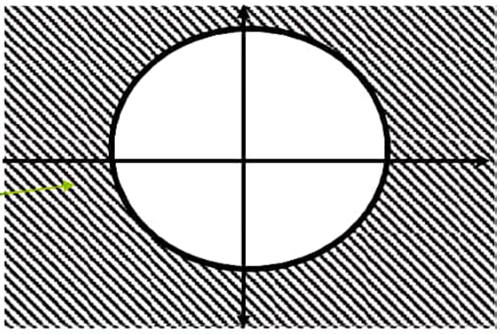
We also know $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ for $|a| < 1$

Thus, x (z) converges when | a z⁻¹ | < 1

$$X(z) = \frac{1}{1-az^{-1}}$$ for | a z⁻¹ | < 1

for | a /z | < 1

$$X(z) = \frac{z}{z-a}$$ for | z | > |a|



ROC is outside the circle |z|=|a|

Solution(3)

$$x(n) = a^n U(n) + b^n U(-n-1)$$

Sequence is non-causal

Z.T. for the given sequence x (n) is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{n=-\infty}^{-1} (bz^{-1})^n$$

Put n=-m in second term

$$\infty \qquad n=0 \qquad \sum_{m=1}^{\infty} (b^{-1} z)^m$$

We know that $\sum_{n=0}^{\infty} a^n = a^0 + a^1 + a^2 + ...$

Series converges iff |a|<1

We also know $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ for $|a| < 1$

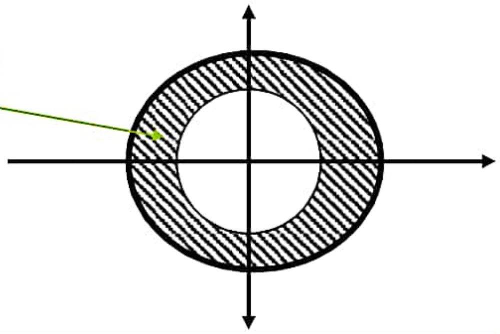
Thus, x (z) converges when | a z⁻¹ | < 1 & |b⁻¹z|<1

$$X(z) = \frac{\square 1}{1-az^{-1}} + \frac{b^{-1}z}{1-b^{-1}z}$$ for | a z⁻¹ | < 1 & | b⁻¹ z | <1

for | a /z | < 1 & |z/b| <1

$$X(z) = \frac{z}{z-a} + \frac{z}{b-z}$$ for | z | > |a| & | z | < |b|

ROC is |b|>| z | > |a|



Solution(1)

$$\boxed{x(n) = 7\left(\tfrac{1}{3}\right)^n u(n) - 6\left(\tfrac{1}{2}\right)^n u(n)}$$

$$X(z) = 7\sum_{n=0}^{\infty}\left(\tfrac{1}{3}\right)^n z^{-n} - 6\sum_{n=0}^{\infty}\left(\tfrac{1}{2}\right)^n z^{-n}$$

$$X(z) = 7\sum_{n=0}^{\infty}\left(\tfrac{1}{3} z^{-1}\right)^n - 6\sum_{n=0}^{\infty}\left(\tfrac{1}{2} z^{-1}\right)^n$$

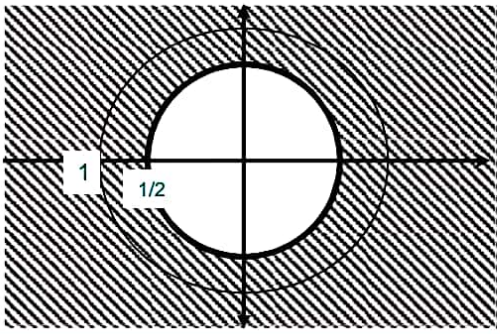
$$= \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}}$$

$$\left|\tfrac{1}{3}z^{-1}\right| < 1 \quad \& \quad \left|\tfrac{1}{2}z^{-1}\right| < 1$$

$$|z| > \tfrac{1}{3} \quad \& \quad |z| > \tfrac{1}{2}$$

$$= \frac{7\square_2 - \frac{7}{3}z^{-1} - 6 + \frac{6}{}\, z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}$$

ROC is
outside the
circle |z|=1/2



$$= \frac{z\square_2^2\left(1+\left(-\frac{7}{3}+\frac{6}{}\right)z^{-1}\right)}{(z-\frac{1}{3})(z-\frac{1}{2})} \qquad = \frac{z\,\square(z_2 - \frac{3}{})}{(z-\frac{1}{3})(z-\frac{1}{2})}$$

Solution(2)

$$\boxed{x[n] = \left(\tfrac{1}{3}\right)^n \sin\left(\tfrac{\pi}{4} n\right) u[n]}$$

$$x[n] = \left(\tfrac{1}{3}\right)^n \left[\frac{e^{j\pi/4 n} - e^{-j\pi/4 n}}{2j} \right] u[n] \quad = \frac{1}{2j}\left(\frac{1}{3}\right)^n e^{j\,\pi/4 n} u[n] - \frac{1}{2j}\left(\frac{1}{3}\right)^n e^{-j\pi/4 n} u[n]$$

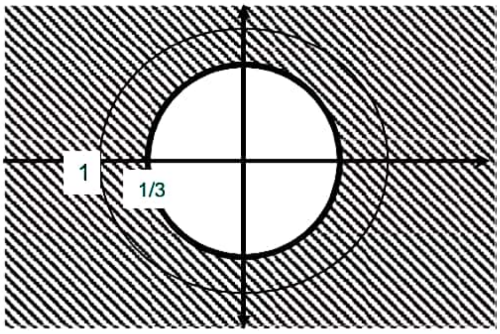
$$X(z) = \frac{1}{2j} \frac{1}{1 - \tfrac{1}{3} e^{j\pi/4 n} z^{-1}} - \frac{1}{2j} \frac{1}{1 - \tfrac{1}{3} e^{-j\pi/4 n} z^{-1}}$$

ROC is outside the circle |z|=1/3

ROC $\left|\frac{1}{3} e^{j\pi/4 n} z^{-1}\right| < 1 \,\&\, \left|\frac{1}{3} e^{-j\pi/4 n} z^{-1}\right|$

$|z| > 1/3$ & $|z| > 1/3$

$$|z| > 1/3$$



Solution(3)

$$\boxed{x(n) = b^{|n|} \qquad b>0}$$

$$x(n) = b^{n}u(n) + b^{-n}u(-n-1)$$

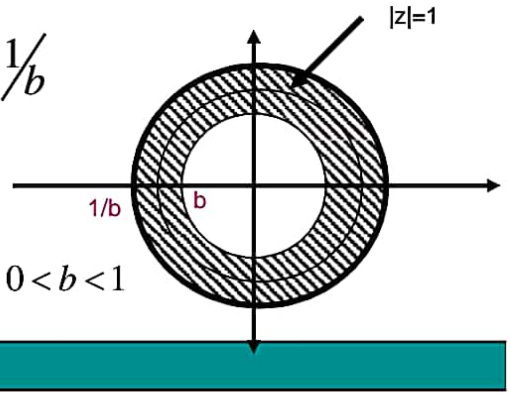
We know

$$b^{n}u[n] \Leftrightarrow \frac{\square 1}{1-bz^{-1}} \qquad |z|>b$$

and $$b^{-n}u[-n-1] \Leftrightarrow \frac{\square 1}{1-b^{-1}z^{-1}} \qquad |z|<1/b$$

$$X(z) = \frac{1}{1-bz^{-1}} - \frac{1}{1-b^{-1}z^{-1}} \qquad b<|z|<1/b$$

For b>1, there are no values of z that satisfy ROC

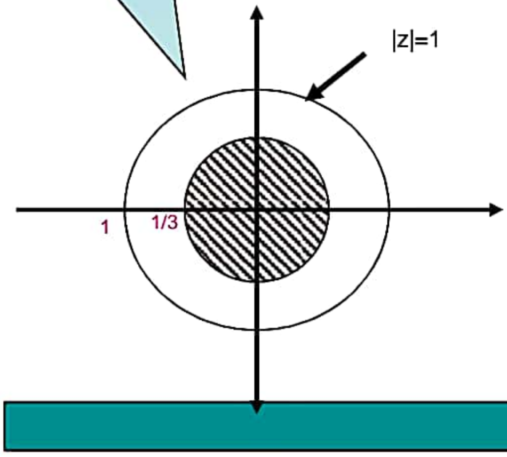


Problem:

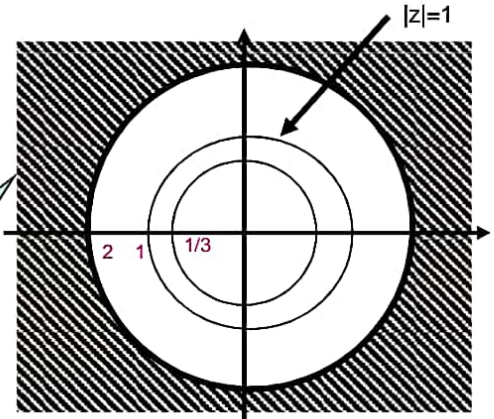
Show all possible ROC's and pole-zero diagram z-transform given below

$$X(z) = \frac{1}{(1-\tfrac{1}{3}z^{-1})(1-2z^{-1})}$$

If x[n] is left sided signal
i.e. anti-causal signal



If x[n] is right sided signal
i.e. causal signal



If x[n] is double sided signal
i.e. non-causal signal