Region of Convergence (ROC)

(Z-Transforms)

Objective : To understand the meaning of ROC in Z transforms and the need to consider it.

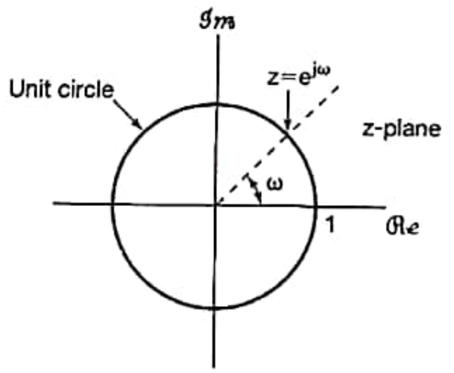
Introduction :

As we are aware that the Z- transform of a discrete signal *x(n)* is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

The Z-transform has two parts which are, the expression and Region of Convergence respectively.

Whether the Z-transform X(z) of a signal x(n) exists or not depends on the complex variable 'z' as well as the signal itself. All complex values of '$z=re^{j\omega}$' for which the summation in the definition converges form a *region of convergence (ROC)* in the z-plane. A circle with r=1 is called unit circle and the complex variable in z-plane is represented as shown below.



Description :

The concept of ROC can be understood easily by finding z transform of two functions given below:

a) $x(n) = a^n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n)z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence of X(z), we require that $\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$. Consequently, the region of convergence is that range of values of z for which $|az^{-1}| < 1$, or equivalently, $|z| > |a|$ and is shown in figure below

Roots of numerator polynomial are called zeros and the roots of denominator polynomial are called poles. Poles in z-plane are indicated with 'x' and zeros with'o' similar to s-plane. The representation of X(z) through its poles and zeros in the z-plane is referred to as the *pole-zero plot* of X(z).

In general, we assume the order of the numerator polynomial is always lower than that of the denominator polynomial, i.e., M<N. If this is not the case, we can always expand X(z) into multiple terms so that M<N is true for each of terms.

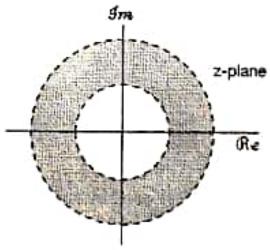
Properties of ROC:

In Module-15 we saw that there were specific properties of the region of convergence of the Laplace transform for different classes of signals and that understanding these properties led to further insights about the transform. In a similar manner, we explore a number of properties of the region of convergence for the z-transform

Property 1:*The ROC of X(z) consists of a ring in the z-plane centered about the origin.*

This property is illustrated in figure below and follows from the fact that the ROC consists of those values of $z=re^{j\omega}$ for which $x(n)r^{-n}$ has a Fourier transform that converges. That is, the ROC of the z-transform of x(n) consists of the values of z for which $x(n)r^{-n}$ is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |x(n)| r^{-n} < \infty$$

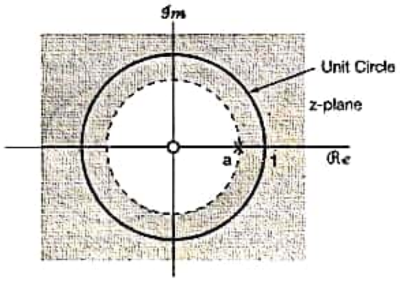


Thus, convergence is dependent only on r=|z| and not on ω. Consequently, if a specific value of z is in the ROC, then all values of z on the same circle (i.e., with the same magnitude) will be in ROC. This by itself guarantees that ROC will consist of concentric rings.

In some cases, the inner boundary of the ROC may extend inward to the origin, and in some cases the outer boundary may extend outward to infinity.

Property 2:*If the z-transform X(z) of x(n) is rational, then the ROC does not contain any poles but is bounded by poles or extend to infinity.*

As with the Laplace transform, this property is simply a consequence of the fact that at a pole X(z) is infinite and therefore does not converge.



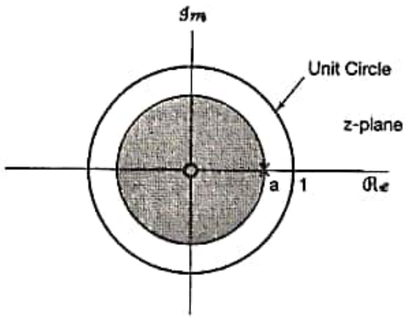
Then $X(z) = \sum_{n=0}^{\infty}(az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$

b) $x(n) = -a^n u(-n-1)$

$$X(z) = \sum_{n=-\infty}^{\infty}\{-a^n u(-n-1)\}z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty}(a^{-1}z)^n$$

$$= 1 - \sum_{n=0}^{\infty}(a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

This result converges only when $|a^{-1}z| < 1$, or equivalently, $|z| < |a|$. The ROC is shown below



Need to consider region of convergence while determining the z-transform

If we consider the signals $a^n u(n)$ and $-a^n u(-n-1)$, we note that although the signals are differing, their z Transforms are identical which is $\frac{z}{z-a}$. Thus we conclude that to distinguish zTransforms uniquely their ROC's must be specified.

Zeros and Poles of the Laplace Transform

LikeLaplace transforms as studied earlier in Module-15, the z-transforms in the above examples are rational, i.e., they can be written as a ratio of polynomials of variable 'z'in the general form

$$X(z) = \frac{N(z)}{D(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{\prod_{k=1}^{M} z-z_{zk}}{\prod_{k=1}^{N} z-z_{pk}}.$$

- *N(z)* is the numerator polynomial of order M withz_{zk},(k=1,2,…,M) roots
- *D(z)* is the denominator polynomial of order N with z_{pk}(k=1,2,…,N) roots

***Property 3**: If x(n) is of finite duration, then the ROC is the entire z-plane, except possibly z=0 and / or z=∞*

A finite duration sequence has only a finite number of nonzero values, extending, say, from n=N to n=M, where N and M are finite. Thus the z-transform is the sum of a finite number of terms; that is

$$X(z) = \sum_{n=N}^{M} x(n)z^{-n}$$

For z not equal to zero or infinity, each term in the sum will be finite, and consequently X(z) will converge.

If N is negative and M is positive, so that x(n) has nonzero values both for n<0 and n>0, then the summation includes terms with both positive and negative powers of z. As |z|→0, terms involving negative powers of z, become unbounded, and as |z|→∞, terms involving positive powers of z become unbounded. Consequently, for N negative and M positive, the ROC does not include z=0 or z=∞.

If N is zero or positive, there are only negative powers of z and consequently, the ROC includes z=∞. If M is zero or negative, there are only positive powers of z and consequently, the ROC includes z=0.

Illustration

Consider the unit impulse signal $\delta(n)$. Its z-transform is given by

$$\delta(n) \overset{Z}{\leftrightarrow} \sum_{n=-\infty}^{\infty} \delta(n)z^{-n} = 1$$

with an ROC consisting of the entire z-plane, including z=0 and z=∞. On the other hand, consider the delayed unit impulse $\delta(n-1)$, for which

$$\delta(n-1) \overset{Z}{\leftrightarrow} \sum_{n=-\infty}^{\infty} \delta(n-1)z^{-n} = z^{-1}$$

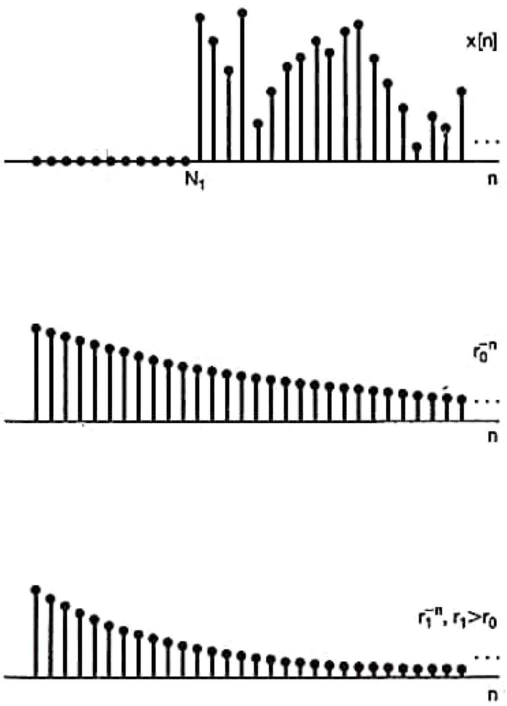
This z-transform is well defined except at z=0 where there is a pole. Thus, the ROC consists of the entire z-plane, including z=∞ but excluding z=0. Similarly, consider an impulse advanced in time by one unit, namely $\delta(n+1)$.

$$\delta(n+1) \overset{Z}{\leftrightarrow} \sum_{n=-\infty}^{\infty} \delta(n+1)z^{-n} = z$$

In this case, ROC consists of the entire finite z-plane (including z=0), but there is a pole at infinity.

Property 4: *If x(nt) is a right sided sequence, and if the circle $|z|=r_o$ is in the ROC, then all finite values of z for which $|z|>r_o$ will also be in the ROC.*

The justification for this property follow in a manner identical to that in Laplace transforms. A right sided sequence is zero prior to some value of n, say N_1. If the circle $|z|=r_o$ is in the ROC, then $x(n)r^{-n}$ is absolutely summable. Now consider $|z|=r_1$ with $r_1>r_o$, so that r_1^{-n} decays quickly than r_o^{-n} for increasing n as illustrated in the figure below.



Consequently, $x(n)r_1^{-n}$ is absolutely summable.

For right sided sequences in general $X(z) = \sum_{n=N1}^{\infty} x(n)z^{-n}$, where N_1 is finite and may be positive or negative.

If N_1 is negative, then the summation above includes terms with positive powers of z, which become unbounded as $|z|\rightarrow\infty$. Consequently, for right sided sequences in general, ROC will not include infinity.

However, for causal sequences, i.e., sequences that are zero for n<0, N_1 will be non-negative, and consequently, the ROC will include $z=\infty$

Property 5: *If x(n) is a left sided sequence, and if the circle $|z|=r_o$ is in the ROC, then all values of z for which $0<|z|<r_o$ will also be in the ROC.*

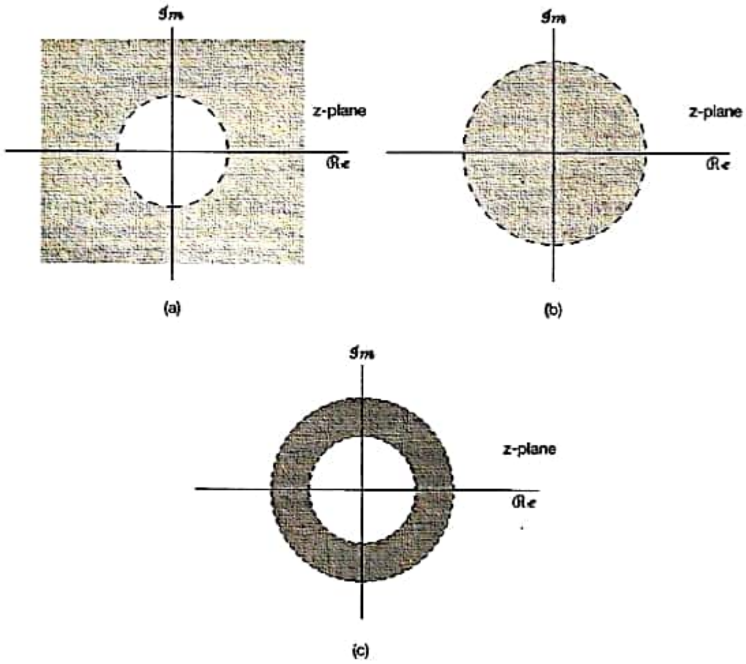
For left sided sequences, the summation for the z-transform will be of the form

$$x(z) = \sum_{n=-\infty}^{M} x(n)z^{-n}$$

where M may be positive or negative. If M is positive, then the transform includes negative powers of z, which become unbounded as $|z| \to 0$. Consequently, for left-sided sequences, the ROC will not include $|z|=0$. However, if $M \leq 0$ (so that $x(n)=0$ for all $n>0$), the ROC will include $z=0$.

Property 6: *If x(n) is two sided, and if the circle $|z|=r_o$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z|=r_o$.*

Like corresponding property in Laplace transforms, the ROC of a two-sided signal can be examined by expressing x(n) as the sum of a right-sided and a left-sided signal. The ROC for the right-sided component is a region bounded on the inside by a circle and extending outward to (and possibly including) infinity as in figure (a). The ROC for the left-sided component is a region bounded on the outside by a circle and extending inward to, and possibly including, the origin as in figure (b). The ROC for the composite signal includes the intersection of the ROCs of the components as in figure (c).



Property 7: *Ifthe z-transform X(z) of x(n) is rational, and if x(n) is right sided, then the ROC is the region in the z-plane outside the outermost pole i.e., outside the circle of radius equal to the largest magnitude of the poles of X(z).*

Property 8:*If the z-transform X(z) of x(n) is rational, and if x(n) is left sided, then the ROC is the region in the z-plane inside the innermost pole i.e., inside the circle of radius equal to the smallest magnitude of the poles of X(z) other than any at $z=0$ and extending inward to and possibly including $z=0$.*

Analysis and characterization of Discrete Time LTI systems using z- transform:

One of the important applications of the z-transform is in the analysis and characterization of Discrete Time LTI systems.

Causality: For a causal LTI system, the impulse response is zero for $n<0$ and thus is right sided.

A discrete time LTI system with rational system function H(z) is causal if and only if :

(i) The ROC is the exterior of a circle outside the outermost pole

(ii) With H(z) expressed as a ratio of polynomials in z, the order of the numerator cannot be greater than the order of the denominator

Stability: The stability of a Discrete time LTI system is equivalent to its impulse response being absolutely summable.

(i) An LTI system is stable if and only if the ROC of its system function H(z) includes the unit circle [i.e., $|z|=1$]

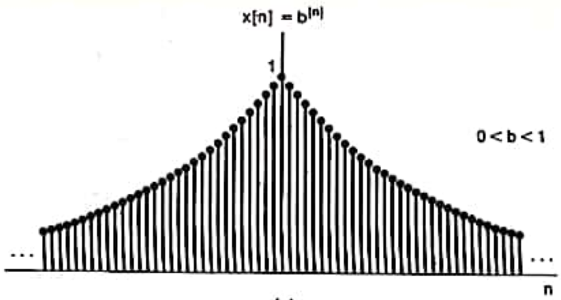
(ii) A causal LTI system with rational transfer function H(s) is stable if and only if all the poles of H(s) lie inside the unit circle – i.e., they must all have magnitude smaller than 1

Examples:

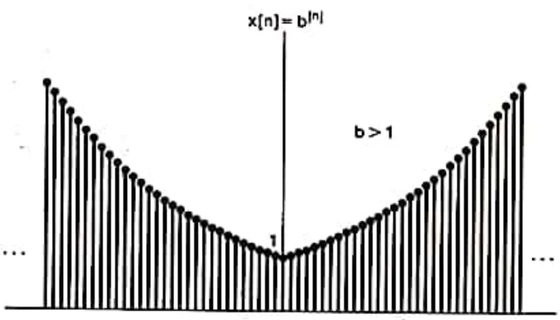
Illustration:Properties6,7 and 8 can be illustrated with an example

$$x(n) = b^{|n|}; b > 0$$

This two sided sequence is illustrated in the figure below:



(a)



(b)

The z-transform for the sequence can be obtained by expressing it as the sum of a right sided and a left sided sequence. We have

$$x(n) = b^n u(n) + b^{-n} u(-n-1)$$

From the examples already studied

$b^n u(n) \overset{Z}{\leftrightarrow} \frac{1}{1-bz^{-1}}; |z| > b$ {right sided}

And $b^{-n} u(-n-1) = \left(\frac{1}{b}\right)^n u(-n-1) \overset{Z}{\leftrightarrow} \frac{-1}{1-\frac{1}{b}z^{-1}}; |z| < \frac{1}{b}$ {left sided}

The Pole Zero plots for the functions are shown below:

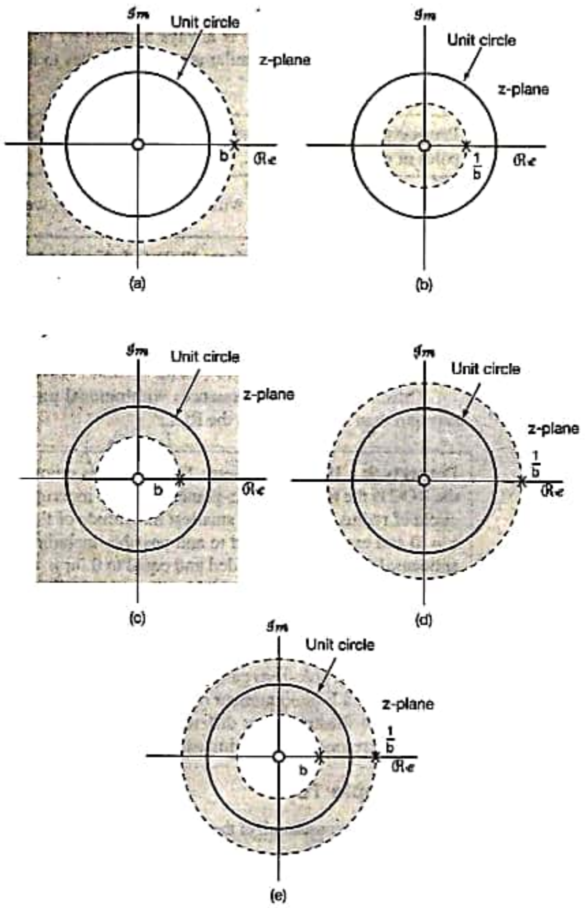


Fig (a) indicates ROC for right sided sequence if b > 1 {causal but not stable}

Fig (b) indicates ROC for left sided sequence if b > 1 {non causal and unstable}

Fig (c) indicates ROC for right sided sequence if 0 <b < 1 { causal and stable}

Fig (d) indicates ROC for left sided sequence if 0 <b < 1 { non causal and stable}

Fig (e) indicates ROC for two sided sequence if 0 < b < 1 {stable}