Discrete–time Fourier Series and Fourier Transforms

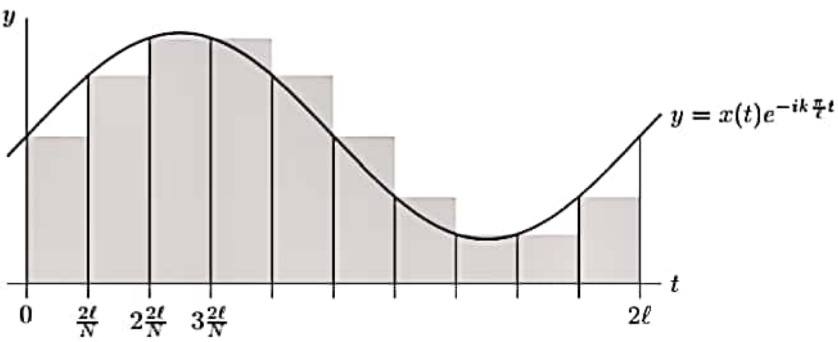
We now start considering discrete–time signals. A discrete–time signal is a function (real or complex valued) whose argument runs over the integers, rather than over the real line. We shall use square brackets, as in $x[n]$, for discrete–time signals and round parentheses, as in $x(t)$, for continuous–time signals. This is the notation used in EECE 359 and EECE 369. Discrete–time signals arise in two ways. Firstly, the signal could really be representing a discrete sequence of values. For example, $x[n]$ could be the $n^{\rm th}$ digit in a string of binary digits being transmitted along some data bus in a computer. Or it could be the maximum temperature for day number n. Secondly, a discrete–time signal could arise from sampling a continuous–time signal at a discrete sequence of times.

Periodic Signals

Just as in the continuous–time case, discrete–time signals may or may not be periodic. We start by considering the periodic case. Imagine an application in which we have to measure some function $x(t)$, that is periodic of period 2ℓ, and compute its Fourier coefficients from the measurements. We can think of $x(t)$ as the amplitude of some periodic signal at time t. Because we can only make finitely many measurements, we cannot determine $x(t)$ for all values of t. Suppose that we measure $x(t)$ at N equally spaced values of t "covering" the full period $0 \le t < 2\ell$. Say at $t = 0, \frac{2\ell}{N}, 2\frac{2\ell}{N}, \cdots, (N-1)\frac{2\ell}{N}$. Because we do not know $x(t)$ for all t we cannot compute the complex[(1)] Fourier coefficient

$$c_k = \frac{1}{2\ell}\int_0^{2\ell} x(t)e^{-ik\frac{\pi}{\ell}t}\,dt \tag{1}$$

exactly. But we can get a Riemann sum approximation to it using only t's for which $x(t)$ is known. All we need to do is divide the domain of integration up into N intervals each of length $\frac{2\ell}{N}$. For t in the



interval $n\frac{2\ell}{N} \le t < (n+1)\frac{2\ell}{N}$, we approximate the integrand $x(t)e^{-ik\frac{\pi}{\ell}t}$ by its value at $t = n\frac{2\ell}{N}$, which is $x\big(n\frac{2\ell}{N}\big)e^{-ik\frac{\pi}{\ell}n\frac{2\ell}{N}} = x\big(n\frac{2\ell}{N}\big)e^{-2\pi i\frac{kn}{N}}$. So we approximate the integral over $n\frac{2\ell}{N} \le t < (n+1)\frac{2\ell}{N}$ by the area of a rectangle of height $x\big(n\frac{2\ell}{N}\big)e^{-2\pi i\frac{kn}{N}}$ and width $\frac{2\ell}{N}$. This gives

$$c_k \approx c_k^{(N)} = \frac{1}{2\ell}\sum_{n=0}^{N-1} x\big(n\tfrac{2\ell}{N}\big)e^{-2\pi i\frac{kn}{N}}\,\frac{2\ell}{N} = \frac{1}{N}\sum_{n=0}^{N-1} x\big(n\tfrac{2\ell}{N}\big)e^{-2\pi i\frac{kn}{N}}$$

To save writing for what follows set $x[n] = x\left(n\frac{2\ell}{N}\right)$ and $\hat{x}[k] = c_k^{(N)}$. Then

$$\hat{x}[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-2\pi i\frac{kn}{N}}$$

Note that $x[n]$ and $\hat{x}[k]$ are both periodic of period N. That is

$$\begin{aligned}\hat{x}[k+N] &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-2\pi i\frac{(k+N)n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-2\pi i\frac{kn}{N}}e^{-2\pi i\frac{Nn}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-2\pi i\frac{kn}{N}} \text{ because } e^{-2\pi ni} = 1\\ &= \hat{x}[k]\\ x[n+N] &= x\left((n+N)\tfrac{2\ell}{N}\right) = x\left(n\tfrac{2\ell}{N} + 2\ell\right) = x\left(n\tfrac{2\ell}{N}\right) = x[n]\end{aligned}$$

The vector $\left(\hat{x}[k]\right)_{k=0,1,2,\cdots,N-1}$, defined by,

$$\hat{x}[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-2\pi i\frac{kn}{N}} \tag{2}$$

is called the discrete Fourier series (or by some people the discrete Fourier transform) of the vector $\left(x[j]\right)_{j=0,1,2,\cdots,N-1}$. One of the main facts about discrete Fourier series is that we can recover all of the (N different) $x[n]$'s *exactly* from $\hat{x}[0]$, $\hat{x}[1]$, $\cdots$, $\hat{x}[N-1]$ (or any other N consecutive $\hat{x}[k]$'s) using the inverse formula

$$x[n] = \sum_{k=0}^{N-1} \hat{x}[k]\, e^{2\pi i\frac{nk}{N}} \tag{3}$$

Proof: We need to show that if $\hat{x}[k]$ is defined by (2), then (3) is true. To verify this we just substitute the definition of $\hat{x}[k]$ into the right hand side of (3), taking care to rename the summation variable to ensure that we don't use n to stand for two different quantities in the same formula.

$$\begin{aligned}\sum_{k=0}^{N-1} e^{2\pi i\frac{nk}{N}}\hat{x}[k] &= \sum_{k=0}^{N-1} e^{2\pi i\frac{nk}{N}}\frac{1}{N}\sum_{n'=0}^{N-1} e^{-2\pi i\frac{n'k}{N}}x[n'] = \frac{1}{N}\sum_{n'=0}^{N-1}\sum_{k=0}^{N-1} e^{2\pi i\frac{k(n-n')}{N}}x[n']\\ &= \frac{1}{N}\sum_{n'=0}^{N-1} x[n']\sum_{k=0}^{N-1} e^{2\pi i\frac{k(n-n')}{N}} \quad = \frac{1}{N}\sum_{n'=0}^{N-1} x[n']\sum_{k=0}^{N-1}\left(e^{2\pi i\frac{n-n'}{N}}\right)^k\\ &= \frac{1}{N}\sum_{n'=0}^{N-1} x[n']\sum_{k=0}^{N-1} r^k \quad \text{with } r = e^{2\pi i\frac{n-n'}{N}}\end{aligned}$$

For one value of n', namely $n' = n$, $r = 1$ and

$$\sum_{k=0}^{N-1} e^{2\pi i\frac{k(n-n')}{N}} = \sum_{k=0}^{N-1} 1 = N$$

For all other values of n', we can use the standard formula

$$1 + r + r^2 + \cdots r^p = \frac{1 - r^{p+1}}{1 - r}$$

(which you can check by multiplying out $(1-r)(1+r+\cdots+r^p)$ and getting $1-r^{p+1}$) with $p=N-1$ and $r=e^{2\pi i\frac{(n-n')}{N}}$ to get

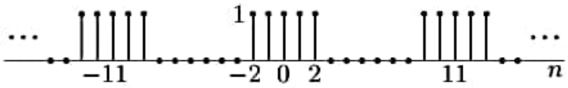
$$\sum_{k=0}^{N-1} e^{2\pi i\frac{k(n-n')}{N}} = \sum_{k=0}^{N-1}\left(e^{2\pi i\frac{(n-n')}{N}}\right)^k = \frac{1-\left(e^{2\pi i\frac{(n-n')}{N}}\right)^N}{1-e^{2\pi i(n-n')/N}} = \frac{1-e^{2\pi i(n-n')}}{1-e^{2\pi i(n-n')/N}} = 0$$

because $n-n'$ is an integer. Substituting the values we have just found for the k sums gives

$$\begin{aligned}\sum_{k=0}^{N-1} e^{2\pi i\frac{nk}{N}}\hat{x}[k] &= \tfrac{1}{N}\sum_{n'=0}^{N-1} x[n'] \sum_{k=0}^{N-1} e^{2\pi i\frac{k(n-n')}{N}} = \tfrac{1}{N}\sum_{n'=0}^{N-1} x[n']\begin{cases} N & \text{if } n'=n \\ 0 & \text{if } n'\neq n\end{cases} \\ &= x[n]\end{aligned}$$

as desired. ■

Example 1 In this example, we find the Fourier series for the discrete–time periodic square wave shown in the figure



This signal has period $N=11$. In computing its Fourier coefficients, we may sum n over any 11 consecutive values. We choose

$$\hat{x}[k] = \tfrac{1}{N}\sum_{n=-2}^{8} x[n]e^{-2\pi i\frac{kn}{N}} = \tfrac{1}{11}\sum_{n=-2}^{2} e^{-2\pi i\frac{kn}{11}}$$

The sum $\sum_{n=-2}^{2} e^{-2\pi i\frac{kn}{11}}$ is a finite geometric series

$$a+ar+ar^2+\cdots+ar^p = \begin{cases} a\frac{1-r^{p+1}}{1-r} & \text{if } r\neq 1 \\ a(p+1) & \text{if } r=1\end{cases} = \begin{cases} \frac{a-ar^{p+1}}{1-r} & \text{if } r\neq 1 \\ a(p+1) & \text{if } r=1\end{cases}$$

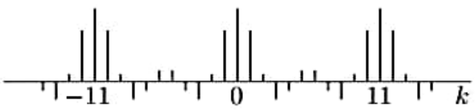
with

- the first term being $a = e^{-2\pi i\frac{kn}{11}}\big|_{n=-2} = e^{4\pi i\frac{k}{11}}$,
- the first "omitted term" being $ar^{p+1} = e^{-2\pi i\frac{kn}{11}}\big|_{n=3} = e^{-6\pi i\frac{k}{11}}$, and
- the ratio between successive terms being $r = e^{-2\pi i\frac{k}{11}}$.

Hence

$$\hat{x}[k] = \frac{1}{11}\frac{e^{4\pi i\frac{k}{11}}-e^{-6\pi i\frac{k}{11}}}{1-e^{-2\pi i\frac{k}{11}}} = \frac{1}{11}\frac{e^{4\pi i\frac{k}{11}}-e^{-6\pi i\frac{k}{11}}}{e^{-\pi i\frac{k}{11}}\left(e^{\pi i\frac{k}{11}}-e^{-\pi i\frac{k}{11}}\right)} = \frac{1}{11}\frac{e^{5\pi i\frac{k}{11}}-e^{-5\pi i\frac{k}{11}}}{2i\sin\left(\pi\frac{k}{11}\right)} = \frac{1}{11}\frac{\sin\left(5\pi\frac{k}{11}\right)}{\sin\left(\pi\frac{k}{11}\right)}$$

provided the ratio $r\neq 1$. That is, provided $k\neq 0,\pm 11,\pm 22,\cdots$. When $k=0,\pm 11,\pm 22,\cdots$, we have that $a=\frac{1}{11}$, $r=1$ and five terms, so that $\hat{x}[k]=\frac{5}{11}$. These Fourier coefficients are graphed in the figure



Both of the sums in (2) and (3) are finite. So there is no problem of truncation error or Gibb's phenomenon when computing discrete Fourier series, at least if N is not humongous.

Discrete–time Fourier series have properties very similar to the linearity, time shifting, etc. properties of the Fourier transform. A table of some of the most important properties is provided at the end of these notes. Here are derivations of a few of them.

Time Shifting: Let n_0 be any integer. If $x[n]$ is a discrete–time signal of period N, then so is $y[n] = x[n - n_0]$. The $k^{\rm th}$ Fourier coefficient of $y[n]$ is

$$\hat{y}[k] = \frac{1}{N}\sum_{n=0}^{N-1} y[n]e^{-2\pi i\frac{kn}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} x[n-n_0]e^{-2\pi i\frac{kn}{N}}$$

Now substitute $m = n - n_0$ in the sum:

$$\hat{y}[k] = \frac{1}{N}\sum_{m=-n_0}^{N-1-n_0} x[m]e^{-2\pi i\frac{k(m+n_0)}{N}} = e^{-2\pi i\frac{kn_0}{N}}\left\{\frac{1}{N}\sum_{m=-n_0}^{N-1-n_0} x[m]e^{-2\pi i\frac{km}{N}}\right\}$$

The summand is periodic of period N. That is, replacing m by $m+N$ has no effect on the summand. So all domains of summation consisting of a single full period give the same sum. Consequently we may replace the sum $\sum_{m=-n_0}^{N-1-n_0}$ by the sum $\sum_{m=0}^{N-1}$ and the sum in parentheses is exactly $\hat{x}[k]$. Thus $\hat{y}[k] = e^{-2\pi i\frac{kn_0}{N}}\hat{x}[k]$.

Conjugation: Notice that if $x[n]$ is a discrete–time signal of period N, then so is $y[n] = \overline{x[n]}$. The $k^{\rm th}$ Fourier coefficient of $y[n]$ is

$$\hat{y}[k] = \frac{1}{N}\sum_{n=0}^{N-1} y[n]\,e^{-2\pi i\frac{nk}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} \overline{x[n]}\,e^{-2\pi i\frac{nk}{N}} = \overline{\frac{1}{N}\sum_{n=0}^{N-1} x[n]\,e^{-2\pi i\frac{n(-k)}{N}}} = \overline{\hat{x}[-k]}$$

This tells us that the $k^{\rm th}$ Fourier coefficient of the periodic discrete–time signal $\overline{x[n]}$ is $\overline{\hat{x}[-k]}$. In particular, $x[n]$ is real valued if and only if $\overline{x[n]} = x[n]$ for all n, which is true if and only the Fourier coefficients of $x[n]$ and $y[n] = \overline{x[n]}$ are the same. That is,

$$x[n] \text{ is real for all } n \iff \overline{\hat{x}[-k]} = \hat{x}[k] \text{ for all } k$$

Parseval's relation: We can derive a version of Parseval's relation for discrete–time Fourier series just as we did for the Fourier transform. Subbing (2) into

$$\begin{aligned}
\sum_{k=0}^{N-1}|\hat{x}[k]|^2 = \sum_{k=0}^{N-1}\{\,\overline{\hat{x}[k]}\,\}\hat{x}[k] \qquad &= \sum_{k=0}^{N-1}\left\{\frac{1}{N}\sum_{n=0}^{N-1}\overline{x[n]}e^{2\pi i\frac{kn}{N}}\right\}\hat{x}[k] \\
= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\overline{x[n]}e^{2\pi i\frac{kn}{N}}\hat{x}[k] &= \frac{1}{N}\sum_{n=0}^{N-1}\overline{x[n]}\left\{\sum_{k=0}^{N-1}e^{2\pi i\frac{kn}{N}}\hat{x}[k]\right\} \\
= \frac{1}{N}\sum_{n=0}^{N-1}\overline{x[n]}\,x[n] \qquad &= \frac{1}{N}\sum_{n=0}^{N-1}\big|x[n]\big|^2
\end{aligned}$$

The Fast Fourier Transform

The "Fast Fourier Transform" does not refer to a new or different type of Fourier transform. It refers to a very efficient algorithm (made popular by a publication of J. W. Cooley and J. W. Tukey in 1965, but actually known to Gauss in about 1805) for computing the discrete–time Fourier and inverse Fourier sums (2) and (3). We will not be covering this algorithm in this course, though it is not particularly sophisticated. The main idea behind it is explained in the supplementary notes "The Fast Fourier Transform".

You have access to the fast Fourier transform through the MATLAB commands `fft` and `ifft`. But a little care must be exercised when using `fft` and `ifft` because they implement different conventions than ours. If the input vector `x` is of length N, then `xhat = fft(x)` is a vector `xhat` with the N elements

$$\texttt{xhat}(k) = \sum_{n=1}^{N} \texttt{x}(n) e^{-2\pi i \frac{(k-1)(n-1)}{N}}, \qquad 1 \le k \le N$$

and if the input vector `xhat` is of length N, then `x = ifft(xhat)` is a vector `x` with the N elements

$$\texttt{x}(n) = \tfrac{1}{N} \sum_{k=1}^{N} \texttt{xhat}(k) e^{2\pi i \frac{(k-1)(n-1)}{N}}, \qquad 1 \le n \le N.$$

In contrast to equations (2) and (3), the factor of $\frac{1}{N}$ appears in `ifft`. The reason for the funny looking exponents is that, in MATLAB, vector indices start with 1 rather than 0.

So given any complex numbers $x[0], \ldots, x[N-1]$, the vector $\hat{x}$ given by equation (2) above can be computed, in MATLAB, as follows:

```
x = [x[0], x[1], x[2], ..., x[N - 1]] ;
xhat = (1/N)*fft(x);
```

Notice the factor of $1/N$ in the second line. The resulting vector `xhat` will have N entries, corresponding to $[\hat{x}[0], \hat{x}[1], \ldots, \hat{x}[N-1]]$. But notice that MATLAB's subscripting rules require

$$\hat{x}[0] = \texttt{xhat}(1), \quad \hat{x}[1] = \texttt{xhat}(2), \quad \cdots, \quad \hat{x}[N-1] = \texttt{xhat}(\texttt{N}).$$

Similarly, if complex numbers $\hat{x}[0], \ldots, \hat{x}[N-1]$ are given, the vector x in equation (3) above can be found using these MATLAB commands:

```
xhat = [x̂[0], x̂[1], ..., x̂[N - 1]] ;
x = N*ifft(xhat);
```

Here, again, the factor of N in the second line corrects for a different system of conventions between these notes and the MATLAB software system.

Aperiodic Signals

We now develop a frequency expansion for non-periodic discrete–time functions using the same strategy as we did in the continuous–time case.

Again, for simplicity we'll only develop the expansions for functions $x[n]$ that are zero for all sufficiently large $|n|$. Again, our conclusions will actually apply to a much broader class of functions. Let N be an even integer that is sufficiently large that $x[n] = 0$ for all $|n| \ge \frac{1}{2}N$. We can get a discrete–time Fourier series expansion for the part of $x[n]$ with $|n| < \frac{1}{2}N$ by using the periodic extension trick. Define $x^{(N)}[n]$ to be the unique discrete–time function determined by the requirements that

$$i)\ x^{(N)}[n] = x[n] \ \text{ for } \ -\tfrac{N}{2} < n \le \tfrac{N}{2}$$
$$ii)\ x^{(N)}[n] \text{ is periodic of period } N$$

Then, for $-\frac{N}{2} < n \le \frac{N}{2}$,

$$x[n] = x^{(N)}[n] = \sum_{-\frac{N}{2} < k \le \frac{N}{2}} \widehat{x^{(N)}}[k]\, e^{2\pi i \frac{nk}{N}} \qquad \text{where} \qquad \widehat{x^{(N)}}[k] = \tfrac{1}{N} \sum_{-\frac{N}{2} < n \le \frac{N}{2}} x[n]\, e^{-2\pi i \frac{nk}{N}} \tag{4}$$

Here we have exploited the fact that since both $x^{(N)}[n]$ and $\widehat{x^{(N)}}[k]$ are periodic of period N, we may choose the range of summation in (2) and (3) to be any consecutive set of N integers. We have chosen $-\frac{N}{2} < k \le \frac{N}{2}$ and $-\frac{N}{2} < n \le \frac{N}{2}$ because they will lead to nice limits when we send $N \to \infty$.

To get a representation of $x[n]$ that is is valid for all n's , not just those in a finite interval $-\frac{N}{2} < n \le \frac{N}{2}$, we take the limit $N \to \infty$. To evaluate this limit we again interpret the sum over k in (4) as a Riemann sum approximation to a certain integral. For each integer k, define the k^{th} frequency to be $\omega_k = 2\pi\frac{k}{N}$. Also use $\Delta\omega = \frac{2\pi}{N}$ to denote the spacing, $\omega_{k+1} - \omega_k$, between any two successive frequencies and define $\hat{x}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n}$. Since $x[n] = 0$ for all $|n| \ge \frac{N}{2}$,

$$\widehat{x^{(N)}}[k] = \frac{1}{N} \sum_{-\frac{N}{2}<n\le\frac{N}{2}} x[n]e^{-2\pi i\frac{nk}{N}} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n]e^{-i2\pi\frac{k}{N}n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega_k n} = \frac{1}{2\pi}\hat{x}(\omega_k)\Delta\omega$$

In this notation,

$$x[n] = x^{(N)}[n] = \sum_{-\frac{N}{2}<k\le\frac{N}{2}} \frac{1}{2\pi}\hat{x}(\omega_k)\Delta\omega\ e^{2\pi i\frac{nk}{N}} = \frac{1}{2\pi} \sum_{\substack{k \text{ with} \\ -\pi<\omega_k\le\pi}} \hat{x}(\omega_k)e^{i\omega_k n}\ \Delta\omega$$

for any $-\frac{N}{2} < n \le \frac{N}{2}$. Note that we have multiplied the summation restriction $-\frac{N}{2} < k \le \frac{N}{2}$ by $\frac{2\pi}{N}$ to get the equivalent restriction $-\pi < \omega_k \le \pi$. As we let $N \to \infty$, the restriction $-\frac{N}{2} < n \le \frac{N}{2}$ disappears and the right hand side, which is exactly a Riemann sum approximation sum to the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{x}(\omega)e^{i\omega n}\,d\omega$, converges to that integral. We conclude that

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{x}(\omega)e^{i\omega n}\,d\omega \qquad \text{where} \qquad \hat{x}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n} \tag{5}$$

The function $\hat{x}(\omega)$ is called the discrete–time Fourier transform of $x[n]$ or the spectrum of $x[n]$. For any integer m,

$$\hat{x}(\omega + 2\pi m) = \sum_{n=-\infty}^{\infty} x[n]e^{-i(\omega+2\pi m)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n}e^{-i2\pi mn} = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n} = \hat{x}(\omega)$$

So $\hat{x}(\omega)$ is periodic of period 2π and we may choose as the domain of integration in (5) any interval of length 2π.

Formulae (5) should look familiar. They are exactly the first Fourier expansion that we saw, but with the roles of the time and frequency domains exchanged. In Theorem 1 of the notes "Fourier Series", we saw that, if $f(t)$ is continuous with continuous first derivative and is also periodic with period 2π, then

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \qquad \text{where} \qquad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,dt$$

If we make the substitutions $t = \omega$ and $k = -n$ we get

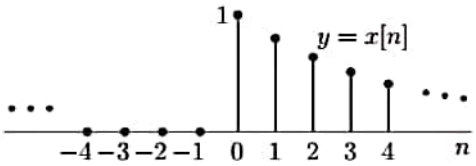
$$f(\omega) = \sum_{n=-\infty}^{\infty} c_{-n}e^{-in\omega} \qquad \text{where} \qquad c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\omega)e^{in\omega}\,d\omega$$

which is exactly (5) with $x[n] = c_{-n}$ and $\hat{x}(\omega) = f(\omega)$.

Example 2 The discrete–time signal

$$x[n] = a^n u[n] \qquad \text{where} \quad u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

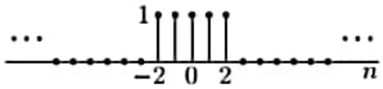
is graphed in the figure



It has the discrete Fourier transform

$$\hat{x}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n} = \sum_{n=0}^{\infty} a^n e^{-i\omega n} = \sum_{n=0}^{\infty} \left(ae^{-i\omega}\right)^n = \frac{1}{1 - ae^{-i\omega}}$$

Example 3 The discrete–time signal shown in the figure



consists of a single pulse from the square wave signal of Example 1. The computation of its Fourier transform is virtually identical to the computation of the Fourier coefficients in Example 1.

$$\hat{x}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n} = \sum_{n=-2}^{2} e^{-i\omega n}$$

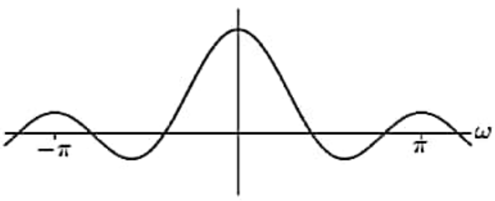
This is again a finite geometric series

$$a + ar + ar^2 + \cdots + ar^p = a\tfrac{1-r^{p+1}}{1-r} \qquad \text{if } r \neq 1$$

this time with first term $a = e^{2i\omega}$, ratio $r = e^{-i\omega}$ and $p = 4$. Hence

$$\hat{x}(\omega) = e^{2i\omega}\frac{1 - e^{-5i\omega}}{1 - e^{-i\omega}} = \frac{e^{\frac{5}{2}i\omega}}{e^{\frac{1}{2}i\omega}}\frac{1 - e^{-5i\omega}}{1 - e^{-i\omega}} = \frac{e^{\frac{5}{2}i\omega} - e^{-\frac{5}{2}i\omega}}{e^{\frac{1}{2}i\omega} - e^{-\frac{1}{2}i\omega}} = \frac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

when $e^{-i\omega} \neq 1$, i.e. when $\omega \neq 2k\pi$, with k an integer. When $\omega = 2k\pi$, we have that $a = 1$ and $r = 1$ so that $\hat{x}(\omega) = 5$. This Fourier transform is graphed in the figure



As sample derivations of the properties of the transform (5), we now develop the two properties that involve convolutions. First suppose that we take the product $p[n] = x[n]y[n]$ of the two discrete–time signals

$x[n]$ and $y[n]$. Then, by (5),

$$\begin{aligned}\hat{p}[\omega] &= \sum_{n=-\infty}^{\infty} x[n]y[n]e^{-i\omega n} = \sum_{n=-\infty}^{\infty} \left\{\tfrac{1}{2\pi}\int_{-\pi}^{\pi} \hat{x}(\theta)e^{i\theta n}\, d\theta\right\} y[n]e^{-i\omega n} \\ &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\ \hat{x}(\theta) \sum_{n=-\infty}^{\infty} y[n]e^{-i(\omega-\theta)n} \\ &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\ \hat{x}(\theta)\hat{y}(\omega-\theta)\end{aligned} \tag{6}$$

which is, aside from the prefactor of $\frac{1}{2\pi}$, the convolution of $\hat{x}$ and $\hat{y}$. Note that the domain of integration is over one period of $\hat{x}$ and $\hat{y}$.

Second, if we take the convolution

$$c[n] = (x * y)[n] = \sum_{m=-\infty}^{\infty} x[n-m]y[m]$$

then

$$\hat{c}(\omega) = \sum_{n=-\infty}^{\infty} c[n]e^{-i\omega n} = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x[n-m]y[m]e^{-i\omega n} = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x[n-m]e^{-i\omega(n-m)}y[m]e^{-i\omega m}$$

For each fixed m

$$\sum_{n=-\infty}^{\infty} x[n-m]e^{-i\omega(n-m)} \overset{k=n-m}{=} \sum_{k=-\infty}^{\infty} x[k]e^{-i\omega k} = \hat{x}(\omega)$$

so

$$\hat{c}(\omega) = \sum_{m=-\infty}^{\infty} \hat{x}(\omega)y[m]e^{-i\omega m} = \hat{x}(\omega)\sum_{m=-\infty}^{\infty} y[m]e^{-i\omega m} = \hat{x}(\omega)\hat{y}(\omega) \tag{7}$$

Example 4 Suppose that we take the convolution of the impulse signal

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

with some other signal $x[n]$. Then

$$(x * \delta)[n] = \sum_{m=-\infty}^{\infty} x[n-m]\delta[m]$$

But, by the definition of δ, every single term in the sum, except that with $m = 0$ is zero. So

$$(x * \delta)[n] = x[n-0]\delta[0] = x[n]$$

By (7), the discrete Fourier transform of $x[n] = (x*\delta)[n]$ is $\hat{x}(\omega)\hat{\delta}(\omega)$. So $\hat{\delta}(\omega)$ ought to be one. By definition,

$$\hat{\delta}(\omega) = \sum_{n=\infty}^{\infty} \delta[n]e^{-i\omega n}$$

Again, by the definition of δ, every term in the sum, except that with $n = 0$, is zero. So $\hat{\delta}(\omega) = \delta[0]e^{-i\omega 0} = 1$ as expected.

Example 5 This time, let's convolve an impulse

$$\delta_{n_0}[n] = \begin{cases} 1 & \text{if } n = n_0 \\ 0 & \text{if } n \neq n_0 \end{cases}$$

at some time n_0, possibly nonzero, with some other signal $x[n]$. Then

$$(x * \delta_{n_0})[n] = \sum_{m=-\infty}^{\infty} x[n-m]\delta_{n_0}[m] = x[n-n_0]\delta_{n_0}[n_0] = x[n-n_0]$$

The DFT of δ_{n_0} is

$$\hat{\delta}_{n_0}(\omega) = \sum_{n=\infty}^{\infty} \delta_{n_0}[n]e^{-i\omega n} = \delta_{n_0}[n_0]e^{-i\omega n_0} = e^{-i\omega n_0}$$

So (7) tells us that the DFT of the time shifted signal $x[n-n_0]$ is

$$\hat{\delta}_{n_0}(\omega)\hat{x}(\omega) = e^{-i\omega n_0}\hat{x}(\omega)$$

Example 6 As a less trivial example of a direct evaluation of a convolution consider the boxcar

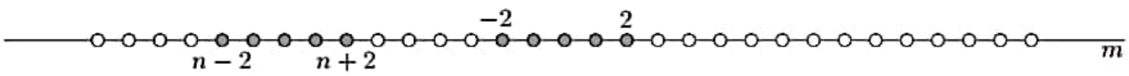
$$\beta[n] = \begin{cases} 1 & \text{if } |n| \le 2 \\ 0 & \text{if } |n| > 2 \end{cases}$$

of Example 3. The convolution

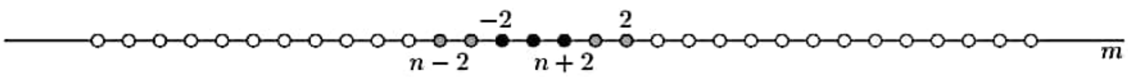
$$(\beta * \beta)[n] = \sum_{m=-\infty}^{\infty} \beta[n-m]\beta[m]$$

The first factor, $\beta[n-m]$ is zero unless $|n-m| \le 2$, i.e. unless m is within distance 2 of n, i.e. unless $n-2 \le m \le n+2$. The second factor, $\beta[m]$ is zero unless $-2 \le m \le 2$. So the convolution $(\beta * \beta)[n]$ is the number of integers m that obey both $n-2 \le m \le n+2$ and $-2 \le m \le 2$.

- If n is so negative that $n+2 < -2$ (i.e. $n < -4$) there are no m's that obey both $n-2 \le m \le n+2$ and $-2 \le m \le 2$. This is illustrated in the figure below. In his case, the convolution is zero.

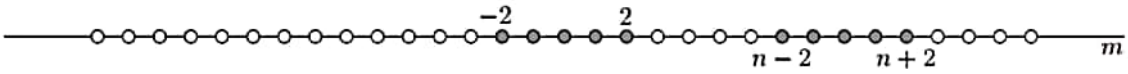


- If we increase n so that $-2 \le n+2 \le 2$ (i.e. $-4 \le n \le 0$) then the inequalities $n-2 \le m \le n+2$ and $-2 \le m \le 2$ reduce to $-2 \le m \le n+2$. This is illustrated in the figure below. In general, if $k \le p$ are two integers, the number of integers m that obey $k \le m \le p$ is $p-k+1$. (In particular, if $p = k$, there is one allowed m and if $p = k+1$, there are two allowed m's.) Applying this with $p = n+2$ and $k = -2$, we have that, in his case, the convolution is $(n+2)-(-2)+1 = n+5$.



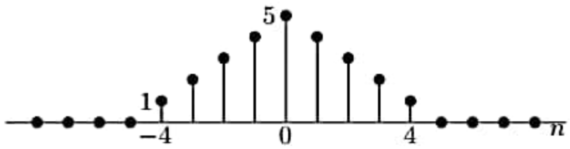
- If we increase n again so that $-2 \le n-2 \le 2$ (i.e. $0 \le n \le 4$) then the inequalities $n-2 \le m \le n+2$ and $-2 \le m \le 2$ reduce to $n-2 \le m \le 2$. This is illustrated in the figure below. In his case the convolution is $2-(n-2)+1 = 5-n$.

- Finally, if n is so positive that $n-2>2$ (i.e. $n>4$) there are no m's that obey both $n-2 \le m \le n+2$ and $-2 \le m \le 2$. This is illustrated in the figure below. In his case the convolution is zero.



We conclude that

$$(\beta * \beta)[n] = \begin{cases} 0 & \text{if } n < -4 \\ n+5 & \text{if } -4 \le n \le 0 \\ 5-n & \text{if } 0 \le n \le 4 \\ 0 & \text{if } n > 4 \end{cases}$$



We have now seen a number of different, but closely related, Fourier expansions. They are given in the following table.

time domain	*frequency domain*	
continuous, period 2ℓ	discrete	$x(t) = \sum_{k=-\infty}^{\infty} \hat{x}[k]\, e^{ik\frac{\pi}{\ell}t}$ $\quad \hat{x}[k] = \frac{1}{2\ell}\int_{-\ell}^{\ell} x(t)\, e^{-ik\frac{\pi}{\ell}t}\, dt$
continuous	continuous	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{x}(\omega)e^{i\omega t}\, d\omega$ $\quad \hat{x}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\, dt$
discrete, period N	discrete, period N	$x[n] = \sum_{k=0}^{N-1} \hat{x}[k]\, e^{2\pi i\frac{nk}{N}}$ $\quad \hat{x}[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-2\pi i\frac{kn}{N}}$
discrete	continuous, period 2π	$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{x}(\omega)e^{i\omega n}\, d\omega$ $\quad \hat{x}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n}$

Observe that

- If in either domain, time or frequency, the function is periodic (not periodic) then the argument in the other domain is discrete (runs continuously).
- If in either domain, time or frequency, the function has a discrete argument (continuous argument), then the transformed function in the other domain is periodic (not periodic).

Not surprisingly, all of four of these transforms have properties very similar to the linearity, time shifting, etc. properties of the Fourier transform. The detailed versions of these properties are given in the following tables.

Continuous–time, period 2ℓ

Property	*Periodic Signal*	*Fourier Coefficients*
	$x(t) = \sum_k \hat{x}[k] e^{ik\frac{\pi}{\ell}t}$ $y(t) = \sum_k \hat{y}[k] e^{ik\frac{\pi}{\ell}t}$	$\hat{x}[k] = \frac{1}{2\ell}\int_{-\ell}^{\ell} x(t) e^{-ik\frac{\pi}{\ell}t}\,dt$ $\hat{y}[k] = \frac{1}{2\ell}\int_{-\ell}^{\ell} y(t) e^{-ik\frac{\pi}{\ell}t}\,dt$
Linearity	$Ax(t) + By(t)$	$A\hat{x}[k] + B\hat{y}[k]$
Time Shifting	$x(t - t_0)$	$e^{-ik\frac{\pi}{\ell}t_0}\hat{x}[k]$
Frequency Shifting	$e^{in\frac{\pi}{\ell}t}x(t)$	$\hat{x}[k-n]$
Conjugation	$\overline{x(t)}$	$\overline{\hat{x}[-k]}$
Time Reversal	$x(-t)$	$\hat{x}[-k]$
Differentiation	$x'(t)$	$ik\frac{\pi}{\ell}\hat{x}[k]$
Convolution	$\int_{-\ell}^{\ell} x(\tau) y(t-\tau)\,d\tau$	$2\ell\,\hat{x}[k]\hat{y}[k]$
Multiplication	$x(t)y(t)$	$\sum_{m=-\infty}^{\infty} \hat{x}[m]\hat{y}[k-m]$
Parseval	$\frac{1}{2\ell}\int_{-\ell}^{\ell} \lvert x(t)\rvert^2\,dt = \sum_{k=-\infty}^{\infty} \lvert\hat{x}[k]\rvert^2$	

Continuous–time, aperiodic

Property	*Aperiodic Signal*	*Fourier Transform*
	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{x}(\omega) e^{i\omega t}\,d\omega$ $y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{y}(\omega) e^{i\omega t}\,d\omega$	$\hat{x}(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t}\,dt$ $\hat{y}(\omega) = \int_{-\infty}^{\infty} y(t) e^{-i\omega t}\,dt$
Linearity	$Ax(t) + By(t)$	$A\hat{x}(\omega) + B\hat{y}(\omega)$
Time Shifting	$x(t - t_0)$	$e^{-i\omega t_0}\hat{x}(\omega)$
Frequency Shifting	$e^{i\omega_0 t}x(t)$	$\hat{x}(\omega - \omega_0)$
Scaling	$x\left(\frac{t}{\alpha}\right)$	$\lvert\alpha\rvert\hat{x}(\alpha\omega)$
Time Shift & Scaling	$x\left(\frac{t-t_0}{\alpha}\right)$	$\lvert\alpha\rvert e^{-i\omega t_0}\hat{x}(\alpha\omega)$
Frequency shift & scaling	$\lvert\alpha\rvert e^{i\omega_0 t}x(\alpha t)$	$\hat{x}\left(\frac{\omega-\omega_0}{\alpha}\right)$
Conjugation	$\overline{x(t)}$	$\overline{\hat{x}(-\omega)}$
Time Reversal	$x(-t)$	$\hat{x}(-\omega)$
t–Differentiation	$x'(t)$	$i\omega\hat{x}(\omega)$
ω–Differentiation	$tx(t)$	$i\frac{d}{d\omega}\hat{x}(\omega)$
Convolution	$\int_{-\infty}^{\infty} x(\tau) y(t-\tau)\,d\tau$	$\hat{x}(\omega)\hat{y}(\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{x}(\theta)\hat{y}(\omega-\theta)\,d\theta$
Duality	$\hat{x}(t)$	$2\pi x(-\omega)$
Parseval	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert\hat{x}(\omega)\rvert^2\,d\omega$	

Discrete-time, period N

Property	*Periodic Signal*	*Fourier Coefficients*
	$x[n] = \sum_{k=0}^{N-1} \hat{x}[k] e^{2\pi i \frac{kn}{N}}$ $y[n] = \sum_{k=0}^{N-1} \hat{y}[k] e^{2\pi i \frac{kn}{N}}$	$\hat{x}[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-2\pi i \frac{kn}{N}}$ $\hat{y}[k] = \frac{1}{N} \sum_{n=0}^{N-1} y[n] e^{-2\pi i \frac{kn}{N}}$
Linearity	$Ax[n] + By[n]$	$A\hat{x}[k] + B\hat{y}[k]$
Time Shifting	$x[n - n_0]$	$e^{-2\pi i \frac{k n_0}{N}} \hat{x}[k]$
Frequency Shifting	$e^{2\pi i \frac{n k_0}{N}} x[n]$	$\hat{x}[k - k_0]$
Conjugation	$\overline{x[n]}$	$\overline{\hat{x}[-k]}$
Time Reversal	$x[-n]$	$\hat{x}[-k]$
Difference	$x[n] - x[n-1]$	$\left(1 - e^{-2\pi i \frac{k}{N}}\right) \hat{x}[k]$
Convolution	$\sum_{m=0}^{N-1} x[m] y[n-m]$	$N\,\hat{x}[k]\hat{y}[k]$
Multiplication	$x[n]y[n]$	$\sum_{m=0}^{N-1} \hat{x}[m]\hat{y}[k-m]$
Duality	$\hat{x}[n]$	$\frac{1}{N} x[-k]$
Parseval	$\frac{1}{N} \sum_{n=0}^{N-1} \|x[n]\|^2 = \sum_{k=0}^{N-1} \|\hat{x}[k]\|^2$	

Discrete-time, aperiodic

Property	*Aperiodic Signal*	*Fourier Transform*
	$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(\omega) e^{i\omega n}\, d\omega$ $y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{y}(\omega) e^{i\omega n}\, d\omega$	$\hat{x}(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-i\omega n}$ $\hat{y}(\omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-i\omega n}$
Linearity	$Ax[n] + By[n]$	$A\hat{x}(\omega) + B\hat{y}(\omega)$
Time Shifting	$x[n - n_0]$	$e^{-i\omega n_0} \hat{x}(\omega)$
Frequency Shifting	$e^{i\omega_0 n} x[n]$	$\hat{x}(\omega - \omega_0)$
Conjugation	$\overline{x[n]}$	$\overline{\hat{x}(-\omega)}$
Time Reversal	$x[-n]$	$\hat{x}(-\omega)$
n–Difference	$x[n] - x[n-1]$	$\left(1 - e^{-i\omega}\right) \hat{x}(\omega)$
ω–Differentiation	$nx[n]$	$i \frac{d}{d\omega} \hat{x}(\omega)$
Convolution	$\sum_{m=-\infty}^{\infty} x[m] y[n-m]$	$\hat{x}(\omega)\hat{y}(\omega)$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(\theta)\hat{y}(\omega - \theta)\, d\theta$
Parseval	$\sum_{n=-\infty}^{\infty} \|x[n]\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \|\hat{x}(\omega)\|^2\, d\omega$	