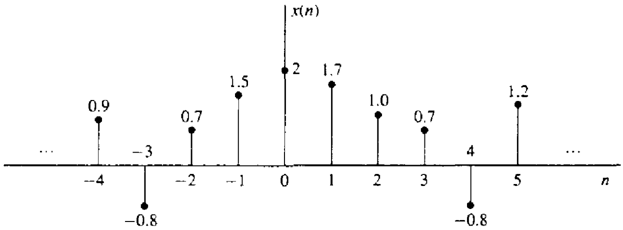


Figure 2.1 Graphical representation of a discrete-time signal.

two successive samples. Also, it is incorrect to think that $x(n)$ is equal to zero if n is not an integer. Simply, the signal $x(n)$ is not defined for noninteger values of n.

In the sequel we will assume that a discrete-time signal is defined for every integer value n for $-\infty < n < \infty$. By tradition, we refer to $x(n)$ as the "nth sample" of the signal even if the signal $x(n)$ is inherently discrete time (i.e., not obtained by sampling an analog signal). If, indeed, $x(n)$ was obtained from sampling an analog signal $x_a(t)$, then $x(n) \equiv x_a(nT)$, where T is the sampling period (i.e., the time between successive samples).

Besides the graphical representation of a discrete-time signal or sequence as illustrated in Fig. 2.1, there are some alternative representations that are often more convenient to use. These are:

1. Functional representation, such as

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases} \tag{2.1.1}$$

2. Tabular representation, such as

n	...	−2	−1	0	1	2	3	4	5	...
$x(n)$	...	0	0	0	1	4	1	0	0	...

3. Sequence representation

An infinite-duration signal or sequence with the time origin ($n = 0$) indicated by the symbol $\uparrow$ is represented as

$$x(n) = \{\ldots 0, \underset{\uparrow}{0}, 1, 4, 1, 0, 0, \ldots\} \tag{2.1.2}$$

A sequence $x(n)$, which is zero for $n < 0$, can be represented as

$$x(n) = \{\underset{\uparrow}{0}, 1, 4, 1, 0, 0, \ldots\} \tag{2.1.3}$$

The time origin for a sequence $x(n)$, which is zero for $n < 0$, is understood to be the first (leftmost) point in the sequence.

A finite-duration sequence can be represented as

$$x(n) = \{3, -1, -2, \underset{\uparrow}{5}, 0, 4, -1\} \tag{2.1.4}$$

whereas a finite-duration sequence that satisfies the condition $x(n) = 0$ for $n < 0$ can be represented as

$$x(n) = \{\underset{\uparrow}{0}, 1, 4, 1\} \tag{2.1.5}$$

The signal in (2.1.4) consists of seven samples or points (in time), so it is called or identified as a seven-point sequence. Similarly, the sequence given by (2.1.5) is a four-point sequence.

2.1.1 Some Elementary Discrete-Time Signals

In our study of discrete-time signals and systems there are a number of basic signals that appear often and play an important role. These signals are defined below.

1. The *unit sample sequence* is denoted as $\delta(n)$ and is defined as

$$\delta(n) \equiv \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases} \tag{2.1.6}$$

   In words, the unit sample sequence is a signal that is zero everywhere, except at $n = 0$ where its value is unity. This signal is sometimes referred to as a *unit impulse*. In contrast to the analog signal $\delta(t)$, which is also called a unit impulse and is defined to be zero everywhere except $t = 0$, and has unit area, the unit sample sequence is much less mathematically complicated. The graphical representation of $\delta(n)$ is shown in Fig. 2.2.

2. The *unit step signal* is denoted as $u(n)$ and is defined as

$$u(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.1.7}$$

   Figure 2.3 illustrates the unit step signal.

3. The *unit ramp signal* is denoted as $u_r(n)$ and is defined as

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.1.8}$$

   This signal is illustrated in Fig. 2.4.

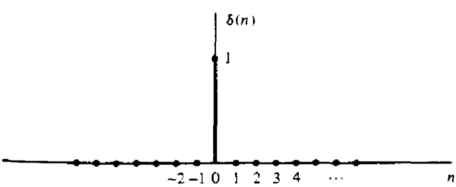


Figure 2.2 Graphical representation of the unit sample signal.

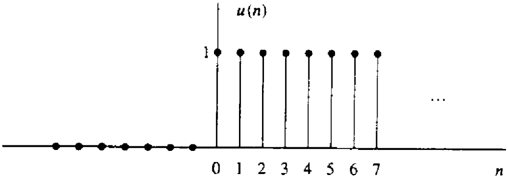


Figure 2.3 Graphical representation of the unit step signal.

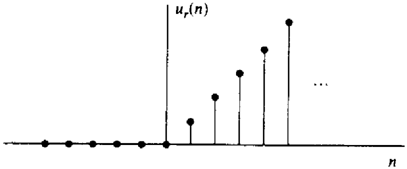


Figure 2.4 Graphical representation of the unit ramp signal.

4. The *exponential signal* is a sequence of the form

$$x(n) = a^n \qquad \text{for all } n \tag{2.1.9}$$

If the parameter a is real, then $x(n)$ is a real signal. Figure 2.5 illustrates $x(n)$ for various values of the parameter a.

When the parameter a is complex valued, it can be expressed as

$$a \equiv re^{j\theta}$$

where r and θ are now the parameters. Hence we can express $x(n)$ as

$$\begin{aligned} x(n) &= r^n e^{j\theta n} \\ &= r^n(\cos\theta n + j\sin\theta n) \end{aligned} \tag{2.1.10}$$

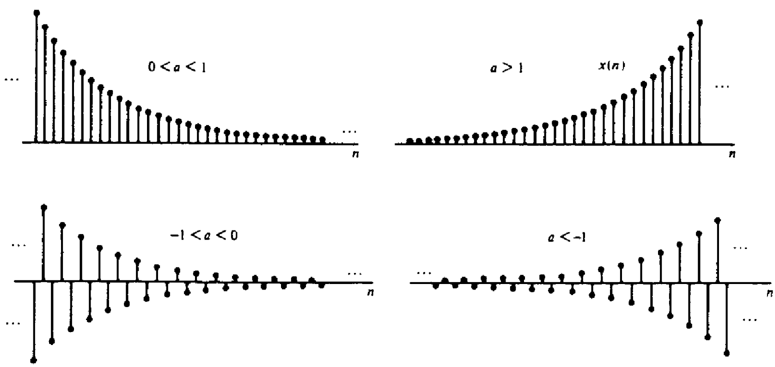


Figure 2.5 Graphical representation of exponential signals.

Since $x(n)$ is now complex valued, it can be represented graphically by plotting the real part

$$x_R(n) \equiv r^n \cos\theta n \tag{2.1.11}$$

as a function of n, and separately plotting the imaginary part

$$x_I(n) \equiv r^n \sin\theta n \tag{2.1.12}$$

as a function of n. Figure 2.6 illustrates the graphs of $x_R(n)$ and $x_I(n)$ for $r = 0.9$ and $\theta = \pi/10$. We observe that the signals $x_R(n)$ and $x_I(n)$ are a damped (decaying exponential) cosine function and a damped sine function. The angle variable θ is simply the frequency of the sinusoid, previously denoted by the (normalized) frequency variable ω. Clearly, if $r = 1$, the damping disappears and $x_R(n)$, $x_I(n)$, and $x(n)$ have a fixed amplitude, which is unity.

Alternatively, the signal $x(n)$ given by (2.1.10) can be represented graphically by the amplitude function

$$|x(n)| = A(n) \equiv r^n \tag{2.1.13}$$

and the phase function

$$\angle x(n) = \phi(n) \equiv \theta n \tag{2.1.14}$$

Figure 2.7 illustrates $A(n)$ and $\phi(n)$ for $r = 0.9$ and $\theta = \pi/10$. We observe that the phase function is linear with n. However, the phase is defined only over the interval $-\pi < \theta \leq \pi$ or, equivalently, over the interval $0 \leq \theta < 2\pi$. Consequently, by convention $\phi(n)$ is plotted over the finite interval $-\pi < \theta \leq \pi$ or $0 \leq \theta < 2\pi$. In other words, we subtract multiples of 2π from $\phi(n)$ before plotting. In one case, $\phi(n)$ is constrained to the range $-\pi < \theta \leq \pi$ and in the other case $\phi(n)$ is constrained to the range $0 \leq \theta < 2\pi$. The subtraction of multiples of 2π from $\phi(n)$ is equivalent to interpreting the function $\phi(n)$ as $\phi(n)$, modulo 2π. The graph for $\phi(n)$, modulo 2π, is shown in Fig. 2.7b.

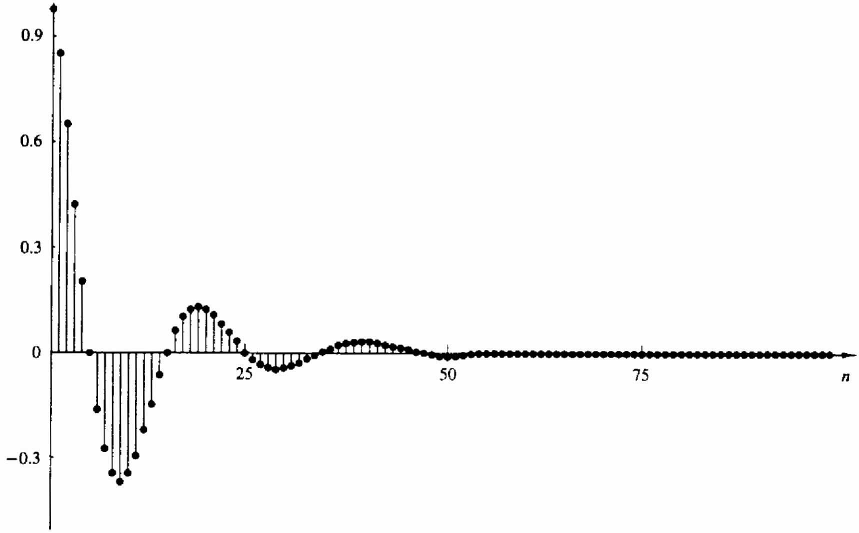
2.1.2 Classification of Discrete-Time Signals

The mathematical methods employed in the analysis of discrete-time signals and systems depend on the characteristics of the signals. In this section we classify discrete-time signals according to a number of different characteristics.

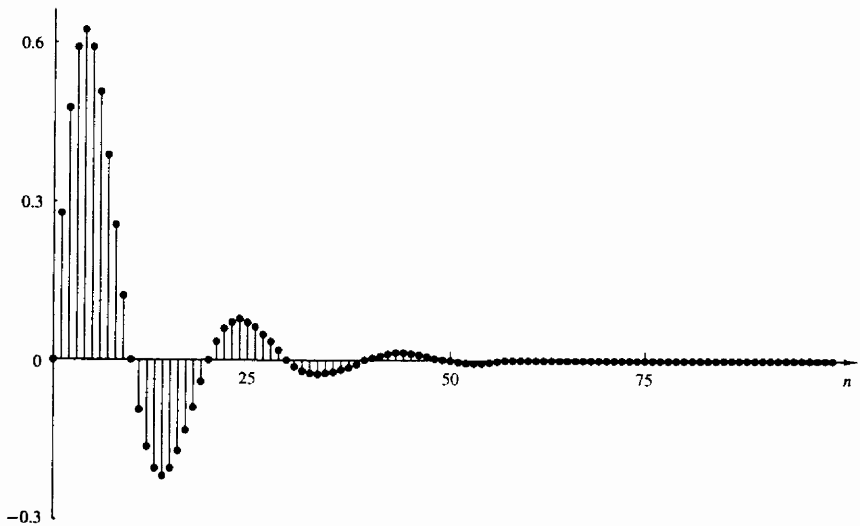
Energy signals and power signals. The energy E of a signal $x(n)$ is defined as

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{2.1.15}$$

We have used the magnitude-squared values of $x(n)$, so that our definition applies to complex-valued signals as well as real-valued signals. The energy of a signal can be finite or infinite. If E is finite (i.e., $0 < E < \infty$), then $x(n)$ is called an *energy*

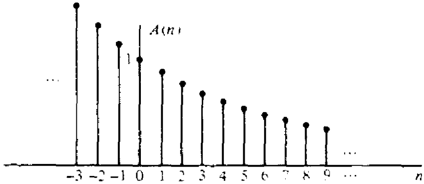


(a) Graph of $x_R(n) = (0.9)^n \cos \frac{\pi n}{10}$

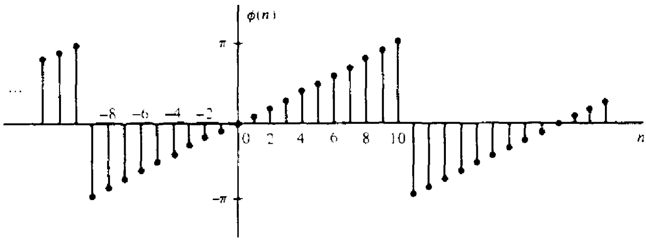


(b) Graph of $x_I(n) = (0.9)^n \sin \frac{\pi n}{10}$

Figure 2.6 Graph of the real and imaginary components of a complex-valued exponential signal.



(a) Graph of $A(n) = r^n$, $r = 0.9$



(b) Graph of $\phi(n) = \frac{\pi}{10}n$, modulo 2π plotted in the range $(-\pi, \pi]$

Figure 2.7 Graph of amplitude and phase function of a complex-valued exponential signal: (a) graph of $A(n) = r^n$, $4 = 0.9$; (b) graph of $\phi(n) = (\pi/10)n$, modulo 2π plotted in the range $(-\pi, \pi]$.

signal. Sometimes we add a subscript x to E and write E_x to emphasize that E_x is the energy of the signal $x(n)$.

Many signals that possess infinite energy, have a finite average power. The average power of a discrete-time signal $x(n)$ is defined as

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \tag{2.1.16}$$

If we define the signal energy of $x(n)$ over the finite interval $-N \leq n \leq N$ as

$$E_N \equiv \sum_{n=-N}^{N} |x(n)|^2 \tag{2.1.17}$$

then we can express the signal energy E as

$$E \equiv \lim_{N\to\infty} E_N \tag{2.1.18}$$

and the average power of the signal $x(n)$ as

$$P \equiv \lim_{N\to\infty} \frac{1}{2N+1} E_N \tag{2.1.19}$$

Clearly, if E is finite, $P = 0$. On the other hand, if E is infinite, the average power P may be either finite or infinite. If P is finite (and nonzero), the signal is called a *power signal*. The following example illustrates such a signal.

Example 2.1.1

Determine the power and energy of the unit step sequence. The average power of the unit step signal is

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} u^2(n)$$

$$= \lim_{N\to\infty} \frac{N+1}{2N+1} = \lim_{N\to\infty} \frac{1+1/N}{2+1/N} = \frac{1}{2}$$

Consequently, the unit step sequence is a power signal. Its energy is infinite.

Similarly, it can be shown that the complex exponential sequence $x(n) = Ae^{j\omega_0 n}$ has average power A^2, so it is a power signal. On the other hand, the unit ramp sequence is neither a power signal nor an energy signal.

Periodic signals and aperiodic signals. As defined on Section 1.3, a signal $x(n)$ is periodic with period $N (N > 0)$ if and only if

$$x(n + N) = x(n) \text{ for all } n \tag{2.1.20}$$

The smallest value of N for which (2.1.20) holds is called the (fundamental) period. If there is no value of N that satisfies (2.1.20), the signal is called *nonperiodic* or *aperiodic*.

We have already observed that the sinusoidal signal of the form

$$x(n) = A \sin 2\pi f_0 n \tag{2.1.21}$$

is periodic when f_0 is a rational number, that is, if f_0 can be expressed as

$$f_0 = \frac{k}{N} \tag{2.1.22}$$

where k and N are integers.

The energy of a periodic signal $x(n)$ over a single period, say, over the interval $0 \le n \le N-1$, is finite if $x(n)$ takes on finite values over the period. However, the energy of the periodic signal for $-\infty \le n \le \infty$ is infinite. On the other hand, the average power of the periodic signal is finite and it is equal to the average power over a single period. Thus if $x(n)$ is a periodic signal with fundamental period N and takes on finite values, its power is given by

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 \tag{2.1.23}$$

Consequently, periodic signals are power signals.

Symmetric (even) and antisymmetric (odd) signals. A real-valued signal $x(n)$ is called symmetric (even) if

$$x(-n) = x(n) \tag{2.1.24}$$

On the other hand, a signal $x(n)$ is called antisymmetric (odd) if

$$x(-n) = -x(n) \tag{2.1.25}$$

We note that if $x(n)$ is odd, then $x(0) = 0$. Examples of signals with even and odd symmetry are illustrated in Fig. 2.8.

We wish to illustrate that any arbitrary signal can be expressed as the sum of two signal components, one of which is even and the other odd. The even signal component is formed by adding $x(n)$ to $x(-n)$ and dividing by 2, that is,

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)] \tag{2.1.26}$$

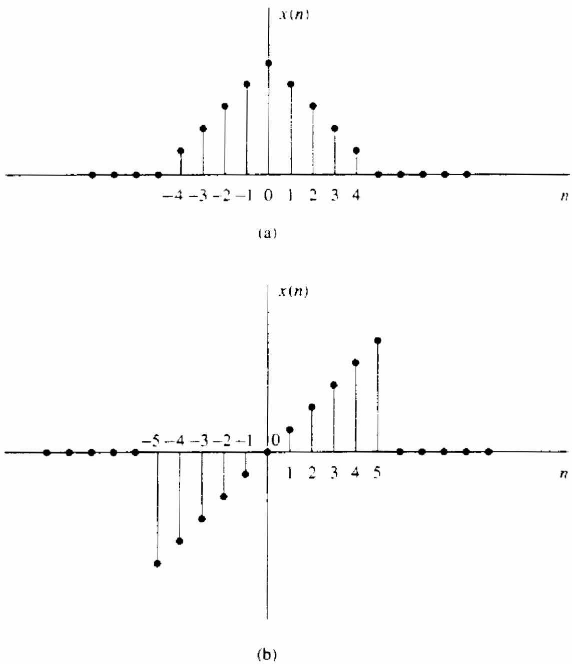


Figure 2.8 Example of even (a) and odd (b) signals.

Clearly, $x_e(n)$ satisfies the symmetry condition (2.1.24). Similarly, we form an odd signal component $x_o(n)$ according to the relation

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)] \tag{2.1.27}$$

Again, it is clear that $x_o(n)$ satisfies (2.1.25); hence it is indeed odd. Now, if we add the two signal components, defined by (2.1.26) and (2.1.27), we obtain $x(n)$, that is,

$$x(n) = x_e(n) + x_o(n) \tag{2.1.28}$$

Thus any arbitrary signal can be expressed as in (2.1.28).

2.1.3 Simple Manipulations of Discrete-Time Signals

In this section we consider some simple modifications or manipulations involving the independent variable and the signal amplitude (dependent variable).

Transformation of the independent variable (time). A signal $x(n)$ may be shifted in time by replacing the independent variable n by $n - k$, where k is an integer. If k is a positive integer, the time shift results in a delay of the signal by k units of time. If k is a negative integer, the time shift results in an advance of the signal by $|k|$ units in time.

Example 2.1.2

A signal $x(n)$ is graphically illustrated in Fig. 2.9a. Show a graphical representation of the signals $x(n-3)$ and $x(n+2)$.

Solution The signal $x(n-3)$ is obtained by delaying $x(n)$ by three units in time. The result is illustrated in Fig. 2.9b. On the other hand, the signal $x(n+2)$ is obtained by advancing $x(n)$ by two units in time. The result is illustrated in Fig. 2.9c. Note that delay corresponds to shifting a signal to the right, whereas advance implies shifting the signal to the left on the time axis.

If the signal $x(n)$ is stored on magnetic tape or on a disk or, perhaps, in the memory of a computer, it is a relatively simple operation to modify the base by introducing a delay or an advance. On the other hand, if the signal is not stored but is being generated by some physical phenomenon in real time, it is not possible to advance the signal in time, since such an operation involves signal samples that have not yet been generated. Whereas it is always possible to insert a delay into signal samples that have already been generated, it is physically impossible to view the future signal samples. Consequently, in real-time signal processing applications, the operation of advancing the time base of the signal is physically unrealizable.

Another useful modification of the time base is to replace the independent variable n by $-n$. The result of this operation is a *folding* or a *reflection* of the signal about the time origin $n = 0$.

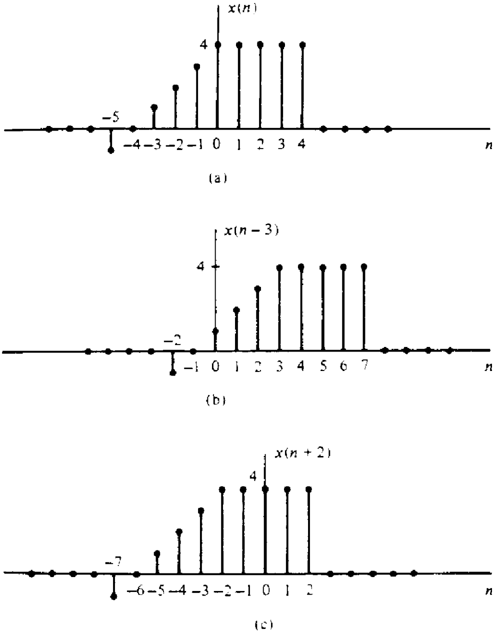


Figure 2.9 Graphical representation of a signal, and its delayed and advanced versions.

Example 2.1.3

Show the graphical representation of the signal $x(-n)$ and $x(-n+2)$, where $x(n)$ is the signal illustrated in Fig. 2.10a.

Solution The new signal $y(n) = x(-n)$ is shown in Fig. 2.10b. Note that $y(0) = x(0)$, $y(1) = x(-1)$, $y(2) = x(-2)$, and so on. Also, $y(-1) = x(1)$, $y(-2) = x(2)$, and so on. Therefore, $y(n)$ is simply $x(n)$ reflected or folded about the time origin $n = 0$. The signal $y(n) = x(-n+2)$ is simply $x(-n)$ delayed by two units in time. The resulting signal is illustrated in Fig. 2.10c. A simple way to verify that the result in Fig. 2.10c is correct is to compute samples, such as $y(0) = x(2)$, $y(1) = x(1)$, $y(2) = x(0)$, $y(-1) = x(3)$, and so on.

It is important to note that the operations of folding and time delaying (or advancing) a signal are not commutative. If we denote the time-delay operation by TD and the folding operation by FD, we can write

$$\begin{aligned} \mathrm{TD}_k[x(n)] &= x(n-k) \qquad k > 0 \\ \mathrm{FD}[x(n)] &= x(-n) \end{aligned} \tag{2.1.29}$$

Now

$$\mathrm{TD}_k\{\mathrm{FD}[x(n)]\} = \mathrm{TD}_k[x(-n)] = x(-n+k) \tag{2.1.30}$$

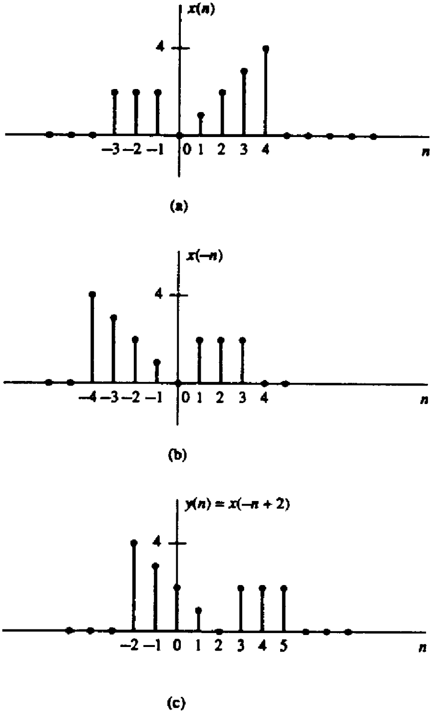


Figure 2.10 Graphical illustration of the folding and shifting operations.

whereas

$$\text{FD}\{\text{TD}_k[x(n)]\} = \text{FD}[x(n-k)] = x(-n-k) \tag{2.1.31}$$

Note that because the signs of n and k in $x(n-k)$ and $x(-n+k)$ are different, the result is a shift of the signals $x(n)$ and $x(-n)$ to the right by k samples, corresponding to a time delay.

A third modification of the independent variable involves replacing n by μn, where μ is an integer. We refer to this time-base modification as *time scaling* or *down-sampling.*

Example 2.1.4

Show the graphical representation of the signal $y(n) = x(2n)$, where $x(n)$ is the signal illustrated in Fig. 2.11a.

Solution We note that the signal $y(n)$ is obtained from $x(n)$ by taking every other sample from $x(n)$, starting with $x(0)$. Thus $y(0) = x(0)$, $y(1) = x(2)$, $y(2) = x(4)$, ... and $y(-1) = x(-2)$, $y(-2) = x(-4)$, and so on. In other words, we have skipped

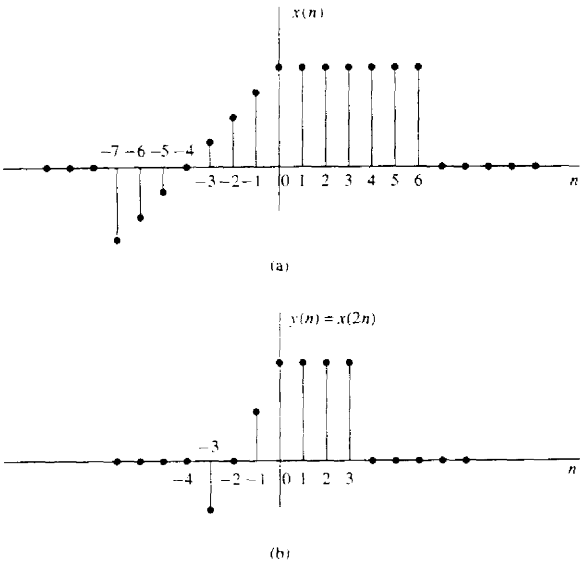


Figure 2.11 Graphical illustration of down-sampling operation.

the odd-numbered samples in $x(n)$ and retained the even-numbered samples. The resulting signal is illustrated in Fig. 2.11b.

If the signal $x(n)$ was originally obtained by sampling an analog signal $x_a(t)$, then $x(n) = x_a(nT)$, where T is the sampling interval. Now, $y(n) = x(2n) = x_a(2Tn)$. Hence the time-scaling operation described in Example 2.1.4 is equivalent to changing the sampling rate from $1/T$ to $1/2T$, that is, to decreasing the rate by a factor of 2. This is a *downsampling* operation.

Addition, multiplication, and scaling of sequences. Amplitude modifications include *addition, multiplication,* and *scaling* of discrete-time signals.

Amplitude scaling of a signal by a constant A is accomplished by multiplying the value of every signal sample by A. Consequently, we obtain

$$y(n) = Ax(n) \qquad -\infty < n < \infty$$

The *sum* of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$, whose value at any instant is equal to the sum of the values of these two signals at that instant, that is,

$$y(n) = x_1(n) + x_2(n) \qquad -\infty < n < \infty$$

The *product* of two signals is similarly defined on a sample-to-sample basis as

$$y(n) = x_1(n)x_2(n) \qquad -\infty < n < \infty$$

2.2 DISCRETE-TIME SYSTEMS

In many applications of digital signal processing we wish to design a device or an algorithm that performs some prescribed operation on a discrete-time signal. Such a device or algorithm is called a discrete-time system. More specifically, a *discrete-time system* is a device or algorithm that operates on a discrete-time signal, called the *input* or *excitation*, according to some well-defined rule, to produce another discrete-time signal called the *output* or *response* of the system. In general, we view a system as an operation or a set of operations performed on the input signal $x(n)$ to produce the output signal $y(n)$. We say that the input signal $x(n)$ is *transformed* by the system into a signal $y(n)$, and express the general relationship between $x(n)$ and $y(n)$ as

$$y(n) \equiv \mathcal{T}[x(n)] \tag{2.2.1}$$

where the symbol $\mathcal{T}$ denotes the transformation (also called an operator), or processing performed by the system on $x(n)$ to produce $y(n)$. The mathematical relationship in (2.2.1) is depicted graphically in Fig. 2.12.

There are various ways to describe the characteristics of the system and the operation it performs on $x(n)$ to produce $y(n)$. In this chapter we shall be concerned with the time-domain characterization of systems. We shall begin with an input–output description of the system. The input–output description focuses on the behavior at the terminals of the system and ignores the detailed internal construction or realization of the system. Later, in Section 7.5, we introduce the state-space description of a system. In this description we develop mathematical equations that not only describe the input–output behavior of the system but specify its internal behavior and structure.

2.2.1 Input–Output Description of Systems

The input–output description of a discrete-time system consists of a mathematical expression or a rule, which explicitly defines the relation between the input and output signals (*input–output relationship*). The exact internal structure of the system is either unknown or ignored. Thus the only way to interact with the system is by using its input and output terminals (i.e., the system is assumed to be a "black box" to the user). To reflect this philosophy, we use the graphical representa-

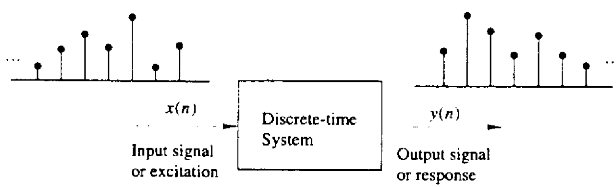


Figure 2.12 Block diagram representation of a discrete-time system.

tion depicted in Fig. 2.12, and the general input–output relationship in (2.2.1) or, alternatively, the notation

$$x(n) \xrightarrow{\mathcal{T}} y(n) \tag{2.2.2}$$

which simply means that $y(n)$ is the response of the system $\mathcal{T}$ to the excitation $x(n)$. The following examples illustrate several different systems.

Example 2.2.1

Determine the response of the following sytems to the input signal

$$x(n) = \begin{cases} |n|, & -3 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

(a) $y(n) = x(n)$

(b) $y(n) = x(n-1)$

(c) $y(n) = x(n+1)$

(d) $y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$

(e) $y(n) = max\{x(n+1), x(n), x(n-1)\}$

(f) $y(n) = \sum_{k=-\infty}^{n} x(k) = x(n) + x(n-1) + x(n-2) + \cdots$ (2.2.3)

Solution First, we determine explicitly the sample values of the input signal

$$x(n) = \{\ldots, 0, 3, 2, 1, \underset{\uparrow}{0}, 1, 2, 3, 0, \ldots\}$$

Next, we determine the output of each system using its input–output relationship.

(a) In this case the output is exactly the same as the input signal. Such a system is known as the *identity* system.

(b) This system simply delays the input by one sample. Thus its output is given by

$$x(n) = \{\ldots, 0, 3, 2, \underset{\uparrow}{1}, 0, 1, 2, 3, 0, \ldots\}$$

(c) In this case the system "advances" the input one sample into the future. For example, the value of the output at time $n = 0$ is $y(0) = x(1)$. The response of this system to the given input is

$$x(n) = \{\ldots, 0, 3, 2, 1, 0, \underset{\uparrow}{1}, 2, 3, 0, \ldots\}$$

(d) The output of this system at any time is the mean value of the present, the immediate past, and the immediate future samples. For example, the output at time $n = 0$ is

$$y(0) = \tfrac{1}{3}[x(-1) + x(0) + x(1)] = \tfrac{1}{3}[1 + 0 + 1] = \tfrac{2}{3}$$

Repeating this computation for every value of n, we obtain the output signal

$$y(n) = \{\ldots, 0, 1, \tfrac{5}{3}, 2, 1, \underset{\uparrow}{\tfrac{2}{3}}, 1, 2, \tfrac{5}{3}, 1, 0, \ldots\}$$

(e) This system selects as its output at time n the maximum value of the three input samples $x(n-1)$, $x(n)$, and $x(n+1)$. Thus the response of this system to the input signal $x(n)$ is

$$y(n) = \{0, 3, 3, \underset{\uparrow}{3}, 2, 1, 2, 3, 3, 3, 0, \ldots\}$$

(f) This system is basically an *accumulator* that computes the running sum of all the past input values up to present time. The response of this system to the given input is

$$y(n) = \{\ldots, 0, 3, 5, 6, \underset{\uparrow}{6}, 7, 9, 12, 0, \ldots\}$$

We observe that for several of the systems considered in Example 2.2.1 the output at time $n = n_0$ depends not only on the value of the input at $n = n_0$ [i.e., $x(n_0)$], but also on the values of the input applied to the system before and after $n = n_0$. Consider, for instance, the accumulator in the example. We see that the output at time $n = n_0$ depends not only on the input at time $n = n_0$, but also on $x(n)$ at times $n = n_0 - 1$, $n_0 - 2$, and so on. By a simple algebraic manipulation the input–output relation of the accumulator can be written as

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{n} x(k) = \sum_{k=-\infty}^{n-1} x(k) + x(n) \\ &= y(n-1) + x(n) \end{aligned} \tag{2.2.4}$$

which justifies the term *accumulator*. Indeed, the system computes the current value of the output by adding (accumulating) the current value of the input to the previous output value.

There are some interesting conclusions that can be drawn by taking a close look into this apparently simple system. Suppose that we are given the input signal $x(n)$ for $n \geq n_0$, and we wish to determine the output $y(n)$ of this system for $n \geq n_0$. For $n = n_0, n_0 + 1, \ldots$, (2.2.4) gives

$$\begin{aligned} y(n_0) &= y(n_0 - 1) + x(n_0) \\ y(n_0 + 1) &= y(n_0) + x(n_0 + 1) \end{aligned}$$

and so on. Note that we have a problem in computing $y(n_0)$, since it depends on $y(n_0 - 1)$. However,

$$y(n_0 - 1) = \sum_{k=-\infty}^{n_0 - 1} x(k)$$

that is, $y(n_0 - 1)$ "summarizes" the effect on the system from all the inputs which had been applied to the system before time n_0. Thus the response of the system for $n \geq n_0$ to the input $x(n)$ that is applied at time n_0 is the combined result of this input and all inputs that had been applied previously to the system. Consequently, $y(n)$, $n \geq n_0$ is not uniquely determined by the input $x(n)$ for $n \geq n_0$.

The additional information required to determine $y(n)$ for $n \geq n_0$ is the *initial condition* $y(n_0 - 1)$. This value summarizes the effect of all previous inputs to the system. Thus the initial condition $y(n_0 - 1)$ together with the input sequence $x(n)$ for $n \geq n_0$ uniquely determine the output sequence $y(n)$ for $n \geq n_0$.

If the accumulator had no excitation prior to n_0, the initial condition is $y(n_0 - 1) = 0$. In such a case we say that the system is *initially relaxed*. Since $y(n_0 - 1) = 0$, the output sequence $y(n)$ depends only on the input sequence $x(n)$ for $n \geq n_0$.

It is customary to assume that every system is relaxed at $n = -\infty$. In this case, if an input $x(n)$ is applied at $n = -\infty$, the corresponding output $y(n)$ is *solely* and *uniquely* determined by the given input.

Example 2.2.2

The accumulator described by (2.2.3) is excited by the sequence $x(n) = nu(n)$. Determine its output under the condition that:

(a) It is initially relaxed [i.e., $y(-1) = 0$].

(b) Initially, $y(-1) = 1$.

Solution The output of the system is defined as

$$y(n) = \sum_{k=-\infty}^{n} x(k) = \sum_{k=-\infty}^{-1} x(k) + \sum_{k=0}^{n} x(k)$$

$$= y(-1) + \sum_{k=0}^{n} x(k)$$

But

$$\sum_{k=0}^{n} x(k) = \frac{n(n+1)}{2}$$

(a) If the system is initially relaxed, $y(-1) = 0$ and hence

$$y(n) = \frac{n(n+1)}{2} \qquad n \geq 0$$

(b) On the other hand, if the initial condition is $y(-1) = 1$, then

$$y(n) = 1 + \frac{n(n+1)}{2} = \frac{n^2 + n + 2}{2} \qquad n \geq 0$$

2.2.2 Block Diagram Representation of Discrete-Time Systems

It is useful at this point to introduce a block diagram representation of discrete-time systems. For this purpose we need to define some basic building blocks that can be interconnected to form complex systems.

An adder. Figure 2.13 illustrates a system (adder) that performs the addition of two signal sequences to form another (the sum) sequence, which we denote

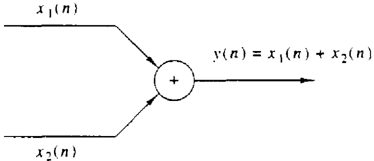


Figure 2.13 Graphical representation of an adder.

as $y(n)$. Note that it is not necessary to store either one of the sequences in order to perform the addition. In other words, the addition operation is *memoryless*.

A constant multiplier. This operation is depicted by Fig. 2.14, and simply represents applying a scale factor on the input $x(n)$. Note that this operation is also memoryless.

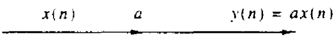


Figure 2.14 Graphical representation of a constant multiplier.

A signal multiplier. Figure 2.15 illustrates the multiplication of two signal sequences to form another (the product) sequence, denoted in the figure as $y(n)$. As in the preceding two cases, we can view the multiplication operation as memoryless.

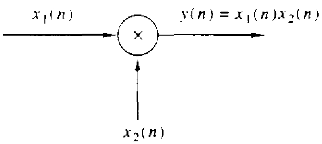


Figure 2.15 Graphical representation of a signal multiplier.

A unit delay element. The unit delay is a special system that simply delays the signal passing through it by one sample. Figure 2.16 illustrates such a system. If the input signal is $x(n)$, the output is $x(n-1)$. In fact, the sample $x(n-1)$ is stored in memory at time $n-1$ and it is recalled from memory at time n to form

$$y(n) = x(n-1)$$

Thus this basic building block requires memory. The use of the symbol z^{-1} to denote the unit of delay will become apparent when we discuss the z-transform in Chapter 3.

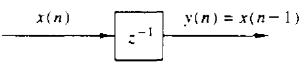


Figure 2.16 Graphical representation of the unit delay element.

A unit advance element. In contrast to the unit delay, a unit advance moves the input $x(n)$ ahead by one sample in time to yield $x(n+1)$. Figure 2.17 illustrates this operation, with the operator z being used to denote the unit advance.

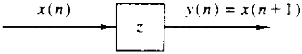


Figure 2.17 Graphical representation of the unit advance element.

We observe that any such advance is physically impossible in real time, since, in fact, it involves looking into the future of the signal. On the other hand, if we store the signal in the memory of the computer, we can recall any sample at any time. In such a nonreal-time application, it is possible to advance the signal $x(n)$ in time.

Example 2.2.3

Using basic building blocks introduced above, sketch the block diagram representation of the discrete-time system described by the input–output relation.

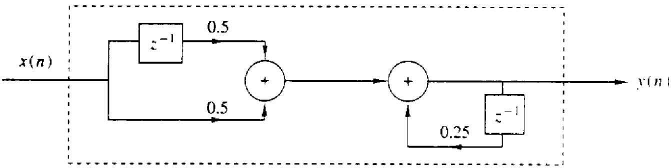
$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}x(n) + \frac{1}{2}x(n-1) \tag{2.2.5}$$

where $x(n)$ is the input and $y(n)$ is the output of the system.

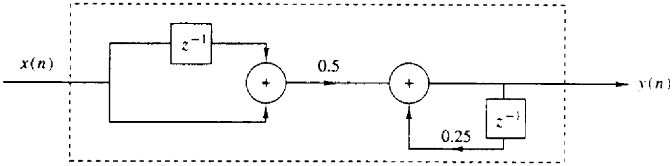
Solution According to (2.2.5), the output $y(n)$ is obtained by multiplying the input $x(n)$ by 0.5, multiplying the previous input $x(n-1)$ by 0.5, adding the two products, and then adding the previous output $y(n-1)$ multiplied by $\frac{1}{4}$. Figure 2.18a illustrates this block diagram realization of the system. A simple rearrangement of (2.2.5), namely,

$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}[x(n) + x(n-1)] \tag{2.2.6}$$

leads to the block diagram realization shown in Fig. 2.18b. Note that if we treat "the system" from the "viewpoint" of an input–output or an external description, we are not concerned about how the system is realized. On the other hand, if we adopt an



(a)



(b)

Figure 2.18 Block diagram realizations of the system $y(n) = 0.25y(n-1) + 0.5x(n) + 0.5x(n-1)$.

internal description of the system, we know exactly how the system building blocks are configured. In terms of such a realization, we can see that a system is *relaxed* at time $n = n_0$ if the outputs of all the *delays* existing in the system are zero at $n = n_0$ (i.e., all memory is *filled* with zeros).

2.2.3 Classification of Discrete-Time Systems

In the analysis as well as in the design of systems, it is desirable to classify the systems according to the general properties that they satisfy. In fact, the mathematical techniques that we develop in this and in subsequent chapters for analyzing and designing discrete-time systems depend heavily on the general characteristics of the systems that are being considered. For this reason it is necessary for us to develop a number of properties or categories that can be used to describe the general characteristics of systems.

We stress the point that for a system to possess a given property, the property must hold for every possible input signal to the system. If a property holds for some input signals but not for others, the system does not possess that property. Thus a counterexample is sufficient to prove that a system does not possess a property. However, to prove that the system has some property, we must prove that this property holds for every possible input signal.

Static versus dynamic systems. A discrete-time system is called *static* or memoryless if its output at any instant n depends at most on the input sample at the same time, but not on past or future samples of the input. In any other case, the system is said to be *dynamic* or to have memory. If the output of a system at time n is completely determined by the input samples in the interval from $n - N$ to $n (N \geq 0)$, the system is said to have *memory* of duration N. If $N = 0$, the system is static. If $0 < N < \infty$, the system is said to have *finite memory*, whereas if $N = \infty$, the system is said to have *infinite memory*.

The systems described by the following input–output equations

$$y(n) = ax(n) \tag{2.2.7}$$

$$y(n) = nx(n) + bx^3(n) \tag{2.2.8}$$

are both static or memoryless. Note that there is no need to store any of the past inputs or outputs in order to compute the present output. On the other hand, the systems described by the following input–output relations

$$y(n) = x(n) + 3x(n-1) \tag{2.2.9}$$

$$y(n) = \sum_{k=0}^{n} x(n-k) \tag{2.2.10}$$

$$y(n) = \sum_{k=0}^{\infty} x(n-k) \tag{2.2.11}$$

are dynamic systems or systems with memory. The systems described by (2.2.9) and (2.2.10) have finite memory, whereas the system described by (2.2.11) has infinite memory.

We observe that static or memoryless systems are described in general by input–output equations of the form

$$y(n) = \mathcal{T}[x(n), n] \tag{2.2.12}$$

and they do not include delay elements (memory).

Time-invariant versus time-variant systems. We can subdivide the general class of systems into the two broad categories, time-invariant systems and time-variant systems. A system is called time-invariant if its input–output characteristics do not change with time. To elaborate, suppose that we have a system $\mathcal{T}$ in a relaxed state which, when excited by an input signal $x(n)$, produces an output signal $y(n)$. Thus we write

$$y(n) = \mathcal{T}[x(n)] \tag{2.2.13}$$

Now suppose that the same input signal is delayed by k units of time to yield $x(n-k)$, and again applied to the same system. If the characteristics of the system do not change with time, the output of the relaxed system will be $y(n-k)$. That is, the output will be the same as the response to $x(n)$, except that it will be delayed by the same k units in time that the input was delayed. This leads us to define a time-invariant or shift-invariant system as follows.

Definition. A relaxed system $\mathcal{T}$ is *time invariant* or *shift invariant* if and only if

$$x(n) \xrightarrow{\mathcal{T}} y(n)$$

implies that

$$x(n-k) \xrightarrow{\mathcal{T}} y(n-k) \tag{2.2.14}$$

for every input signal $x(n)$ and every time shift k.

To determine if any given system is time invariant, we need to perform the test specified by the preceding definition. Basically, we excite the system with an arbitrary input sequence $x(n)$, which produces an output denoted as $y(n)$. Next we delay the input sequence by same amount k and recompute the output. In general, we can write the output as

$$y(n, k) = \mathcal{T}[x(n-k)]$$

Now if this output $y(n, k) = y(n-k)$, for all possible values of k, the system is time invariant. On the other hand, if the output $y(n, k) \neq y(n-k)$, even for one value of k, the system is time variant.

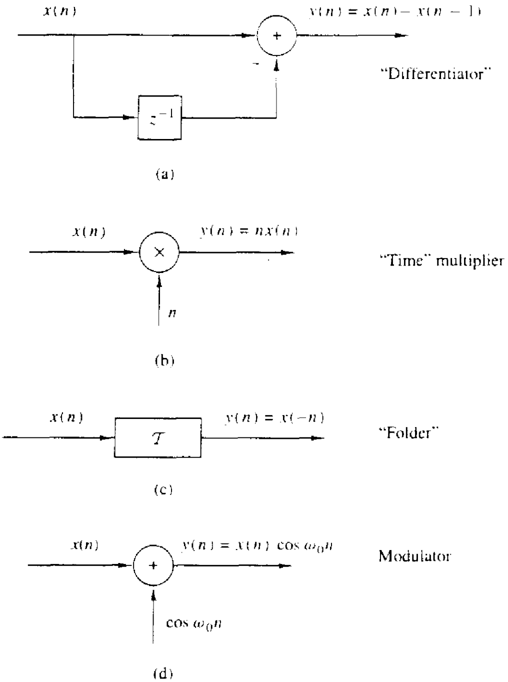


Figure 2.19 Examples of a time-invariant (a) and some time-variant systems (b)–(d).

Example 2.2.4

Determine if the systems shown in Fig. 2.19 are time invariant or time variant.

Solution

(a) This system is described by the input–output equations

$$y(n) = \mathcal{T}[x(n)] = x(n) - x(n-1) \tag{2.2.15}$$

Now if the input is delayed by k units in time and applied to the system, it is clear from the block diagram that the output will be

$$y(n, k) = x(n-k) - x(n-k-1) \tag{2.2.16}$$

On the other hand, from (2.2.14) we note that if we delay $y(n)$ by k units in time, we obtain

$$y(n-k) = x(n-k) - x(n-k-1) \tag{2.2.17}$$

Since the right-hand sides of (2.2.16) and (2.2.17) are identical, it follows that $y(n, k) = y(n-k)$. Therefore, the system is time invariant.

(b) The input–output equation for this system is

$$y(n) = \mathcal{T}[x(n)] = nx(n) \tag{2.2.18}$$

The response of this system to $x(n-k)$ is

$$y(n, k) = nx(n-k) \tag{2.2.19}$$

Now if we delay $y(n)$ in (2.2.18) by k units in time, we obtain

$$\begin{aligned} y(n-k) &= (n-k)x(n-k) \\ &= nx(n-k) - kx(n-k) \end{aligned} \tag{2.2.20}$$

This system is time variant, since $y(n, k) \neq y(n-k)$.

(c) This system is described by the input–output relation

$$y(n) = \mathcal{T}[x(n)] = x(-n) \tag{2.2.21}$$

The response of this system to $x(n-k)$ is

$$y(n, k) = \mathcal{T}[x(n-k)] = x(-n-k) \tag{2.2.22}$$

Now, if we delay the output $y(n)$, as given by (2.2.21), by k units in time, the result will be

$$y(n-k) = x(-n+k) \tag{2.2.23}$$

Since $y(n, k) \neq y(n-k)$, the system is time variant.

(d) The input–output equation for this system is

$$y(n) = x(n) \cos \omega_0 n \tag{2.2.24}$$

The response of this system to $x(n-k)$ is

$$y(n, k) = x(n-k) \cos \omega_0 n \tag{2.2.25}$$

If the expression in (2.2.24) is delayed by k units and the result is compared to (2.2.25), it is evident that the system is time variant.

Linear versus nonlinear systems. The general class of systems can also be subdivided into linear systems and nonlinear systems. A linear system is one that satisfies the *superposition principle*. Simply stated, the principle of superposition requires that the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of the responses (outputs) of the system to each of the individual input signals. Hence we have the following definition of linearity.

Definition. A relaxed $\mathcal{T}$ system is linear if and only if

$$\mathcal{T}[a_1 x_1(n) + a_2 x_2(n)] = a_1 \mathcal{T}[x_1(n)] + a_2 \mathcal{T}[x_2(n)] \tag{2.2.26}$$

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$, and any arbitrary constants a_1 and a_2.

Figure 2.20 gives a pictorial illustration of the superposition principle.

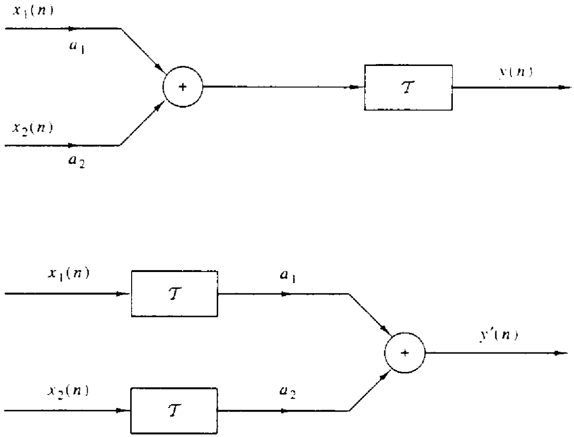


Figure 2.20 Graphical representation of the superposition principle. $\mathcal{T}$ is linear if and only if $y(n) = y'(n)$.

The superposition principle embodied in the relation (2.2.26) can be separated into two parts. First, suppose that $a_2 = 0$. Then (2.2.26) reduces to

$$\mathcal{T}[a_1 x_1(n)] = a_1 \mathcal{T}[x_1(n)] = a_1 y_1(n) \tag{2.2.27}$$

where

$$y_1(n) = \mathcal{T}[x_1(n)]$$

The relation (2.2.27) demonstrates the *multiplicative* or *scaling property* of a linear system. That is, if the response of the system to the input $x_1(n)$ is $y_1(n)$, the response to $a_1 x_1(n)$ is simply $a_1 y_1(n)$. Thus any scaling of the input results in an identical scaling of the corresponding output.

Second, suppose that $a_1 = a_2 = 1$ in (2.2.26). Then

$$\begin{aligned} \mathcal{T}[x_1(n) + x_2(n)] &= \mathcal{T}[x_1(n)] + \mathcal{T}[x_1(n)] \\ &= y_1(n) + y_2(n) \end{aligned} \tag{2.2.28}$$

This relation demonstrates the *additivity property* of a linear system. The additivity and multiplicative properties constitute the superposition principle as it applies to linear systems.

The linearity condition embodied in (2.2.26) can be extended arbitrarily to any weighted linear combination of signals by induction. In general, we have

$$x(n) = \sum_{k=1}^{M-1} a_k x_k(n) \xrightarrow{\mathcal{T}} y(n) = \sum_{k=1}^{M-1} a_k y_k(n) \tag{2.2.29}$$

where

$$y_k(n) = \mathcal{T}[x_k(n)] \qquad k = 1, 2, \ldots, M-1 \tag{2.2.30}$$

We observe from (2.2.27) that if $a_1 = 0$, then $y(n) = 0$. In other words, a relaxed, linear system with zero input produces a zero output. If a system produces a nonzero output with a zero input, the system may be either nonrelaxed or nonlinear. If a relaxed system does not satisfy the superposition principle as given by the definition above, it is called *nonlinear*.

Example 2.2.5

Determine if the systems described by the following input–output equations are linear or nonlinear.

(a) $y(n) = nx(n)$ **(b)** $y(n) = x(n^2)$ **(c)** $y(n) = x^2(n)$
(d) $y(n) = Ax(n) + B$ **(e)** $y(n) = e^{x(n)}$

Solution

(a) For two input sequences $x_1(n)$ and $x_2(n)$, the corresponding outputs are

$$\begin{aligned} y_1(n) &= nx_1(n) \\ y_2(n) &= nx_2(n) \end{aligned} \tag{2.2.31}$$

A linear combination of the two input sequences results in the output

$$\begin{aligned} y_3(n) &= \mathcal{T}[a_1x_1(n) + a_2x_2(n)] = n[a_1x_1(n) + a_2x_2(n)] \\ &= a_1nx_1(n) + a_2nx_2(n) \end{aligned} \tag{2.2.32}$$

On the other hand, a linear combination of the two outputs in (2.2.31) results in the output

$$a_1y_1(n) + a_2y_2(n) = a_1nx_1(n) + a_2nx_2(n) \tag{2.2.33}$$

Since the right-hand sides of (2.2.32) and (2.2.33) are identical, the system is linear.

(b) As in part (a), we find the response of the system to two separate input signals $x_1(n)$ and $x_2(n)$. The result is

$$\begin{aligned} y_1(n) &= x_1(n^2) \\ y_2(n) &= x_2(n^2) \end{aligned} \tag{2.2.34}$$

The output of the system to a linear combination of $x_1(n)$ and $x_2(n)$ is

$$y_3(n) = \mathcal{T}[a_1x_1(n) + a_2x_2(n)] = a_1x_1(n^2) + a_2x_2(n^2) \tag{2.2.35}$$

Finally, a linear combination of the two outputs in (2.2.36) yields

$$a_1y_1(n) + a_2y_2(n) = a_1x_1(n^2) + a_2x_2(n^2) \tag{2.2.36}$$

By comparing (2.2.35) with (2.2.36), we conclude that the system is linear.

(c) The output of the system is the square of the input. (Electronic devices that have such an input–output characteristic and are called square-law devices.) From our previous discussion it is clear that such a system is memoryless. We now illustrate that this system is nonlinear.

The responses of the system to two separate input signals are

$$\begin{aligned} y_1(n) &= x_1^2(n) \\ y_2(n) &= x_2^2(n) \end{aligned} \tag{2.2.37}$$

The response of the system to a linear combination of these two input signals is

$$\begin{aligned} y_3(n) &= \mathcal{T}[a_1x_1(n) + a_2x_2(n)] \\ &= [a_1x_1(n) + a_2x_2(n)]^2 \\ &= a_1^2x_1^2(n) + 2a_1a_2x_1(n)x_2(n) + a_2^2x_2^2(n) \end{aligned} \tag{2.2.38}$$

On the other hand, if the system is linear, it would produce a linear combination of the two outputs in (2.2.37), namely,

$$a_1y_1(n) + a_2y_2(n) = a_1x_1^2(n) + a_2x_2^2(n) \tag{2.2.39}$$

Since the actual output of the system, as given by (2.2.38), is not equal to (2.2.39), the system is nonlinear.

(d) Assuming that the system is excited by $x_1(n)$ and $x_2(n)$ separately, we obtain the corresponding outputs

$$\begin{aligned} y_1(n) &= Ax_1(n) + B \\ y_2(n) &= Ax_2(n) + B \end{aligned} \tag{2.2.40}$$

A linear combination of $x_1(n)$ and $x_2(n)$ produces the output

$$\begin{aligned} y_3(n) &= \mathcal{T}[a_1x_1(n) + a_2x_2(n)] \\ &= A[a_1x_1(n) + a_2x_2(n)] + B \\ &= Aa_1x_1(n) + a_2Ax_2(n) + B \end{aligned} \tag{2.2.41}$$

On the other hand, if the system were linear, its output to the linear combination of $x_1(n)$ and $x_2(n)$ would be a linear combination of $y_1(n)$ and $y_2(n)$, that is,

$$a_1y_1(n) + a_2y_2(n) = a_1Ax_1(n) + a_1B + a_2Ax_2(n) + a_2B \tag{2.2.42}$$

Clearly, (2.2.41) and (2.2.42) are different and hence the system fails to satisfy the linearity test.

The reason that this system fails to satisfy the linearity test is not that the system is nonlinear (in fact, the system is described by a linear equation) but the presence of the constant B. Consequently, the output depends on both the input excitation and on the parameter $B \neq 0$. Hence, for $B \neq 0$, the system is not relaxed. If we set $B = 0$, the system is now relaxed and the linearity test is satisfied.

(e) Note that the system described by the input–output equation

$$y(n) = e^{x(n)} \tag{2.2.43}$$

is relaxed. If $x(n) = 0$, we find that $y(n) = 1$. This is an indication that the system is nonlinear. This, in fact, is the conclusion reached when the linearity test, is applied.

Causal versus noncausal systems. We begin with the definition of causal discrete-time systems.

Definition. A system is said to be *causal* if the output of the system at any time n [i.e., $y(n)$] depends only on present and past inputs [i.e., $x(n)$, $x(n-1)$, $x(n-2), \ldots$], but does not depend on future inputs [i.e., $x(n+1)$, $x(n+2), \ldots$]. In mathematical terms, the output of a causal system satisfies an equation of the form

$$y(n) = F[x(n), x(n-1), x(n-2), \ldots] \tag{2.2.44}$$

where $F[\cdot]$ is some arbitrary function.

If a system does not satisfy this definition, it is called *noncausal.* Such a system has an output that depends not only on present and past inputs but also on future inputs.

It is apparent that in real-time signal processing applications we cannot observe future values of the signal, and hence a noncausal system is physically unrealizable (i.e., it cannot be implemented). On the other hand, if the signal is recorded so that the processing is done off-line (nonreal time), it is possible to implement a noncausal system, since all values of the signal are available at the time of processing. This is often the case in the processing of geophysical signals and images.

Example 2.2.6

Determine if the systems described by the following input–output equations are causal or noncausal.

(a) $y(n) = x(n) - x(n-1)$ **(b)** $y(n) = \sum_{k=-\infty}^{n} x(k)$ **(c)** $y(n) = ax(n)$

(d) $y(n) = x(n) + 3x(n+4)$ **(e)** $y(n) = x(n^2)$ **(f)** $y(n) = x(2n)$

(g) $y(n) = x(-n)$

Solution The systems described in parts (a), (b), and (c) are clearly causal, since the output depends only on the present and past inputs. On the other hand, the systems in parts (d), (e), and (f) are clearly noncausal, since the output depends on future values of the input. The system in (g) is also noncausal, as we note by selecting, for example, $n = -1$, which yields $y(-1) = x(1)$. Thus the output at $n = -1$ depends on the input at $n = 1$, which is two units of time into the future.

Stable versus unstable systems. Stability is an important property that must be considered in any practical application of a system. Unstable systems usually exhibit erratic and extreme behavior and cause overflow in any practical implementation. Here, we define mathematically what we mean by a stable system, and later, in Section 2.3.6, we explore the implications of this definition for linear, time-invariant systems.

Definition. An arbitrary relaxed system is said to be bounded input–bounded output (BIBO) stable if and only if every bounded input produces a bounded output.

The conditions that the input sequence $x(n)$ and the output sequence $y(n)$ are bounded is translated mathematically to mean that there exist some finite numbers, say M_x and M_y, such that

$$|x(n)| \le M_x < \infty \qquad |y(n)| \le M_y < \infty \tag{2.2.45}$$

for all n. If, for some bounded input sequence $x(n)$, the output is unbounded (infinite), the system is classified as unstable.

Example 2.2.7

Consider the nonlinear system described by the input–output equation

$$y(n) = y^2(n-1) + x(n)$$

As an input sequence we select the bounded signal

$$x(n) = C\delta(n)$$

where C is a constant. We also assume that $y(-1) = 0$. Then the output sequence is

$$y(0) = C, \quad y(1) = C^2, \quad y(2) = C^4, \quad \ldots, \quad y(n) = C^{2^n}$$

Clearly, the output is unbounded when $1 < |C| < \infty$. Therefore, the system is BIBO unstable, since a bounded input sequence has resulted in an unbounded output.

2.2.4 Interconnection of Discrete-Time Systems

Discrete-time systems can be interconnected to form larger systems. There are two basic ways in which systems can be interconnected: in cascade (series) or in parallel. These interconnections are illustrated in Fig. 2.21. Note that the two interconnected systems are different.

In the cascade interconnection the output of the first system is

$$y_1(n) = \mathcal{T}_1[x(n)] \tag{2.2.46}$$

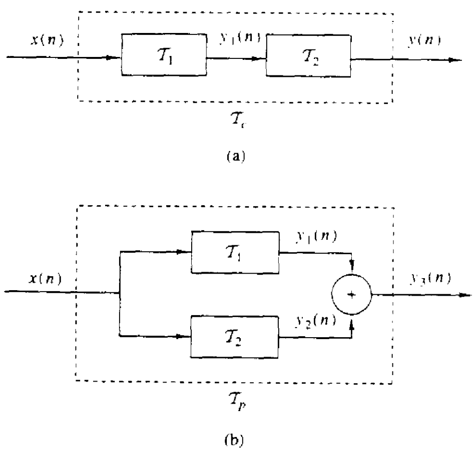


Figure 2.21 Cascade (a) and parallel (b) interconnections of systems.